I examined two contexts of development in children's problem solving: (1) the macro-context of different age cohorts (8-9 vs 11-12 years of age); and (2) the micro-context of an approximately one-hour experimental session. Twenty subjects (even divided across sexes and these two age groups) were individually presented a collection of variously sized gears, a board onto which these gears could be easily attached and rotated, and a knob. Each subject was asked to find all solutions, in which two marked gears were turning the same way, and to represent these solutions graphically. Subjects applied four different problem spaces to the task: the Euclidean, the Kinematic, the Dynamic, and the Topological. The Arithmetic Modifier could be applied to any of these problem spaces, resulting in a numerical characterization of the gear constructions and/or production strategies. The 11-12's tended to shift problem spaces adaptively, the 8-9's seldom did so. Analysis of the pathways of transition between the problem spaces revealed a complex picture of partial or complete incorporation, and substitution.

This research was supported in part by the Spencer Foundation (under a grant entitled "The development of children's understanding of physical phenomena") and in part by the Alfred P. Sloan Foundation (under the Cognitive Science program). I thank Herbert Simon and James Greeno for discussions that clarified both the conceptualization and process of data analysis. David Klahr's reading of the manuscript strengthened its final form. At the project's inception, the suggestions and critical comments of Alex Blanchet and Bärbel Inhelder helped to shape its form and purpose. I am grateful to André Boder, John Clement, David Fallside, George Forman, and Klaus Schultz for questions and criticisms they raised at various stages of the project. I assume full responsibility for the deficiencies that remain. I thank my research assistant, Jacqueline Woolley, for assistance in data coding, preparation of figures, and typing verbal transcripts and most of the manuscript. I also thank Ann Taylor, Director of the Carnegie-Mellon Children's School, for assistance in the location of the school graduates.

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Synopsis of the Task:

The subject is given 12 gears of four different sizes, a Velcro board, and a knob. Each of the gears has a Velcro axis, to enable easy placement anywhere on the board. Two of the gears are marked with the form of a man. When a marked gear is turned clockwise, the man looks like he is somersaulting feet-first. When it is turned counter-clockwise, he looks like he is somersaulting head-first.

E: "Find all the different ways, so that when you turn the knob, both men are doing head-first somersaults."

Figure 1. Materials

C (12.1 years, F)

(2 min into protocol) C has just turned the knob in an arrangement of the two marked gears, with one intervening unmarked gear. Both men have gone feet-first. Instead of simply turning the knob the other direction, she says, "Maybe if I make them closer up, they're going the other way too." S proceeds to change the alignment of two of the gears. She attributes her success to this Euclidean modification.

(8 min into protocol) "Maybe, you know, I can put one the same size in the middle, then it'll work. Maybe if they're all the same size. Yeah! It works!"

(Long pause)

"You know, I think you have to have an odd number in the middle to make it. Like if you put 2 in the middle, one is going to go backwards, 'cause if you only put in one, and then if you put in 3 in the middle, it works. But you can't do it with 4. See if you put 3 in the middle,... I don't think it matters what size it is. I can't do them by themselves because (pause) that's even. See, I found the trick to this. You can only put odd numbers in the middle!"
INTRODUCTION

The Problem

This research investigates the knowledge children employ, in the context of solving a task with gears. The framework for describing the children's knowledge is derived from the work of Artificial Intelligence theorists, McCarthy and Hayes (1969). These authors argue that intelligence is composed of two parts, the epistemological and the heuristic. Each of these components draws heavily upon the other:

The epistemological is the representation of the world in such a form that the solution of problems follows from the facts expressed in the representation. The heuristic part is the mechanism that on the basis of the information solves the problem and decides what to do. (p. 466)

McCarthy and Hayes' framework is readily transposed from the Artificial Intelligence context to the context of children's thinking in physical task domains. I utilize their idea of the epistemological and the heuristic as integrally connected, yet distinct, aspects of knowledge. Their epistemological component can be equated with the representational component of children's thinking. The representational and heuristic together comprise a relatively closed system of the child's viewing and acting upon the world.

The Information Processing construct of problem space is well suited to the conceptualization of this kind of knowledge system. Problem space potentially offers a framework for characterizing both representational and heuristic aspects of cognition during problem solving. As defined by Simon and Hayes (1979):

A problem space is a subject's representation of the task environment that permits him to consider different problem situations, to characterize these situations in ways that may help him decide what to do, and to apply the operators for changing one situation into another. (p. 452)

Problem space has been used as an organizing concept, primarily in research in adult problem solving (Best, 1978; Erikson & Jones, 1978; Keren, 1984; Landweer, 1979; Newell & Simon, 1972; Simon, 1975; Simon & Lea, 1974). Newell (1980) has hypothesized that all symbolic cognition takes place within a problem space. The construct is aptly extended to the development of children's problem solving.

There are two spheres in which problem spaces change: (a) the repertoire of problem spaces across cognitive development, and (b) the problem spaces that are applied to a particular task, during a particular experimental session. Obviously, the problem spaces available to the child at any given point in his or her cognitive development constitute one factor in what prob-
lem spaces are applied. By investigating the problem spaces that different age subjects apply across a relatively extended experimental session, this study addresses both macro- and micro-developments in children’s problem solving.

I consider whether the changes in problem spaces (found in these macro- and micro-contexts) indicate the expansion of a single underlying space, or more complexly related systems. There are two views concerning the relation between successive systems applied to physical task domains. In one view, subsequent approaches involve more complete encoding, which in turn enable more complete processing of relevant information. This kind of sequence can be conceptualized as the expansion of a single problem space. Siegler’s (1981) models of how children, 5 through 12 years of age, make predictions about mechanical equilibrium and projection of shadows constitute examples of this way of thinking. Alternatively, successive systems applied could more closely approximate the complex, nonadditive relation between one paradigm and the next.

In short, I designed the study to address the following questions. First, what problem spaces do children apply in their resolution of a task in an unfamiliar physics task-domain? Second, how often and in what direction do problem space shifts occur, across age and during problem resolution? Third, can the problem spaces applied be viewed as the expansion of a single space? Or do they constitute disjunctive or complexly related systems? Fourth, what are the implications of these results and the micro-details of the protocols for the classic question: “What develops?” (cf. Flavell, 1963; Siegler, 1978).

Selection of the Domain

The research reported here is one study in a series, investigating children’s thinking in the domain of gears. Allen Newell (1973) has suggested that we can work to alleviate the fragmentation characterizing our knowledge of cognition, by focusing a number of studies on a single complex task. Although the delimiter used here is domain as opposed to a particular task, the approach is within the spirit of Newell’s advice. The gears domain meets Newell’s criterion of complexity. Perceptual, spatial, kinematic, dynamic, and mathematical problems arise in this context.

I am not claiming that all children recognize each of these aspects in the domain of gears. Indeed, many children do not treat the task presented in this study as a physics problem at all, but rather as an arbitrary puzzle. Nevertheless, the domain is rich and amenable to experimenter investigation from diverse and complementary angles. There are a number of very different ways for the subject to look at this domain as well. Consequently, it is particularly well-suited to the investigation of problem spaces and problem space shifts.
Analysis of the Task

This particular task explores children's understanding of relative directionality. Subjects seek to build different gear-constructions such that two men, marked on two of the twelve gears, do head-first somersaults. (See Figure 1.) They are also asked to represent graphically the constructions that they evaluate as successful.

Two characteristics of this task differentiate it from most tasks used in the information processing problem-solving research. First, there are no constraints on the search paths. In chess and in puzzles such as the Tower of Hanoi and Missionaries and Cannibals, rules of the game constrain the order in which one can transform a particular state into another. In this task, the placement of one gear only constrains the placements of subsequent gears to the extent that one particular space is at least temporarily blocked. The order of composition, although psychologically significant, is technically irrelevant.

In the problem-solving literature, one solution is typically considered sufficient. The second unusual characteristic of this task is the request to find all the different goal states. Structuring the task in this manner serves several different functions. The experimenter does not need to specify additional tasks. This minimizes post-instructional intervention and, correlative, maximizes the control the subject has over the course of the approximate hour on task. This task characteristic also produces a stable context for potential problem space change. Finally, the manner in which subjects decide what constitutes a "different" solution provides another clue to their problem space.

The simplest and most efficient means of resolving the task involves only one relation: The parity of gear-elements between the marked gears. If there is an odd number of gear-elements between the marked gears, then the two marked gears will turn in the same direction. Conversely, if there is an even number of gear-elements between the marked gears, the marked gears will turn in opposite directions. If there is one connection with an odd number and another connection with an even number, the construction will jam.

Other approaches, such as the abstraction of patterns of relative motion or inference of inversions of directionality across pathways of transmission of movement, offer more adequate physical models of relative directionality, and alternative paths to goal attainment.

Derivation of the Experimental Procedure

I derived the experimental procedures from techniques recently developed in Geneva. Bärbel Inhelder and her research team formulated an experimental strategy, in response to Piaget and Inhelder's concern that cross-age
comparison of levels of thinking was an inadequate means to investigate the 

Inhelder's research team sought to develop tasks, wherein conceptual 
change might be observed. They postulated that a detailed examination of 
the changes in a child's thinking during a single experimental session (30 to 
60 min in length) could show how thinking improves.

According to the Research Team:

The unfolding of the subjects' behavior during such sessions constitutes 
a kind of microgenesis... The investigation of the nature and the ex-
tensiveness of the microgenetic processes at diverse moments of the 
macrogenesis will contribute... to a better understanding of certain fund-
damental mechanisms inherent in development. (Inhelder et al., 1976, 
p. 58) [Translated from the French]

In the prototypical task, the subject is given a collection of concrete 
materials. The only structured aspect of the task is the specification of the 
goal. Tasks are ill-structured, in that the goal-appropriate meaning of the 
elements and the relevant transformations are ambiguous. Different ver-
sions of the goal may be subsequently prescribed. The subject may be asked 
to represent graphically the solutions he or she finds.

The experimenter makes a audio-visual tape of the subject's interac-
tion with the material. Members of the team analyze the tapes collectively, 
on the premise that repeated tape viewing, interspersed with dialogue of 
multiple observers, is the best route to "inter-subjectivity." They argue that 
inter-subjectivity is the closest we can come to objectivity, in the inference 
of the child's evolving meanings (Céllerier, personal communication, 1981). 
For examples of this research, see Ackermann-Valladao (1977), Blanchet 

The modifications of the Genevan procedure as employed here are: (a) 
greater task standardization, (b) use of two independent trained raters and 
one naive rater, (c) the definition of the task in terms of finding all of the 
different solutions, and (d) instructions to think aloud.

The rationale for requesting all of the different solutions has already 
been described. Concerning the instructions to think aloud, many psycholo-
gists profoundly distrust verbal protocols. In a frequently cited work, Nis-
bett and Wilson (1977) analyzed many of the studies using verbal protocols. 
They conclude that verbal protocols often constitute unreliable data. Ericsson 
and Simon (1984) have criticized Nisbett and Wilson for not analyzing the 
conditions of verbal protocol elicitation. Ericsson and Simon also analyzed 
the literature employing verbal protocols, and concluded that verbal proto-
ocols are reliable data under certain conditions.

In this study, I designed the particular timing and form of demands 
for verbalization in accordance with these conditions of reliability. First, 
the request to verbalize is made concurrent with the process of task resolu-
tion. Second, I do not ask the subjects to describe the general process they
are employing, but to verbalize what they are thinking as they work on each
construction. Third, the probes themselves do not suggest any content or in-
ternal representation. Fourth, the task itself is relatively amenable to verbal-
ization.

Derivation of the Problem Space Models and the Coding Schema

Many kinds of analyses confound performances that are identical according
to one or more indices, and yet are distinct in terms of the meaning the sub-
ject attributes to the task domain and strategies he or she employs. For exam-
ple, in the Tower of Hanoi task, Simon (1975) has described four different
strategies, each of which results in the same sequence of moves. Klahr and
Robinson (1981), in their research on some variants of the same task with
preschoolers, identified nine different strategies. Assessing performance
simply in terms of sequence of moves, these authors found that a number of
strategies, including a pair at radically different levels of sophistication,
were “functionally nearly equivalent.”

By using many diverse indices, such as (a) tests carried out in search of
the errors, (b) the nature of the errors, (c) procedures for transforming fail-
ing constructions into successes, (d) procedures for transforming successful
constructions into “different” successes, (e) gesticulations, (f) train of
visual attention, (g) verbal protocol, and (h) properties of the graphic repre-
sentation, we can come much closer to differentiating performances accord-
ing to the problem spaces of the subjects. The task of deriving the problem
space models involves figuring out the meanings implicit in the subjects’
actions and words, and making explicit this complex “web of inference”
(Newell & Simon, 1972) in the form of the coding schema.

The formulation of the problem space models and coding schema was
an interactive process, carried out over the course of three pilot studies.1
The focus continually shifted between definition of kinds of data indicators
and the emerging models themselves.

METHOD

Subjects

Subjects were drawn randomly from the population of graduates of a uni-
versity laboratory preschool, still living in the Pittsburgh area. The popula-
tion is predominately middle and upper-middle socio-economic status (SES).

1 The first pilot was conducted with 15 6–12-year-olds, in a middle SES public school in
Geneva, Switzerland; the second, with 30 6–12-year-olds in a Boston lower-middle SES paro-
chial school; the third, with 17 3–16-year-olds attending or graduates of the university labora-
tory school from which I drew the sample used in this study.
The subjects were ten 8-9-year-olds (range 8.6 to 9.11, mean 9.2) and ten 11-12-year-olds (range 11.5 to 12.3, mean 12.0), with an equal number of males and females in each age group.

Materials

The set of materials given to the subject consists of 12 colored plastic gears, 3 each of 4 different sizes (with diameters 4 cm, 5 cm, 7.5 cm, & 10 cm), a Velcro board, and a knob (Childcraft LOTS-O-GEAR: no. G744). The gears can be easily attached or removed from the board, by means of a Velcro adhesive on the inner circle of the gear-back. This Velcro inner circle also functions as an axis. The edges of the gears have teeth to facilitate interlocking. I marked two of the 7.5 cm gears with a drawing of a man in the course of somersaulting. Turning a marked gear counterclockwise gives the appearance of a man somersaulting feet-first. The knob can be placed in any of 6 of the 12 gears, including both of the marked gears. (See Figure 1.)

Experimental Procedure

I tested each subject individually. The materials are placed on a table in front of the subject.

The initial instructions are:

"There are a bunch of these things [collection of gears indicated]. Two of these have men on them [marked gears indicated]. Lots of them don't. [unmarked gears indicated]. The men can do head-somersaults like this [counter-clockwise, head-first somersault demonstrated]. Or feet-first somersaults like this [clockwise, feet-first somersault demonstrated]. The game is to make something, using any of these things you like, so that when you turn the knob (See, the knob can fit into any of these holes) [turning of a gear with inserted knob demonstrated] both men do head-first somersaults. Please think out loud."

If the subject does not understand the instructions to "think out loud," further explanation is given, for example "Just say whatever you're thinking. Whatever is going through your head, say it."

After the child has built one successful construction, the experimenter extends the task:

"Good! Now the game is to see how many different ways there are to get the two men to do head-first somersaults. Here’s a pen and some paper. Use them to keep track of the ways you find. Remember! It's important to think out loud!"
Beyond these instructions, experimenter intervention is limited to nondirective probes; principally: "Please think out loud," "What are you thinking?," "What are you doing?" "Don't forget to keep track." The subject has a maximum of 60 min to work on the task. The session ends earlier if the subject believes all goal states have been constructed or he or she becomes fatigued. (I do not assume all subjects will find all the solutions in their problem space. In some problem spaces the number of solutions is infinite.)

Procedure of Data Analysis

Dividing the Protocol into Episodes. Each of the two independent coders begin the analysis by dividing the protocol into episodes. The end of an episode is defined in terms of either (a) testing to see if the construction is a success, i.e., turning the knob in a construction the subject considers to be complete or (b) the graphic recording of a construction as another solution, without physical construction and/or without application of the goal test.

Coding the Problem Space(s) of Each Episode. We needed a diversity of indicators for this complex task of differentiating the problem spaces. To determine how the subject represented the task domain, we coded the conceptualization of the gear-constructions he or she built. The indicators used for this purpose were: (a) spontaneous labeling of elements and relations; material, spatial, kinematic and/or abstract (b) verbal and/or gestural representation of gear-constructions (c) types of construction-variations that the subject accepts as forming "different" constructions; and (d) properties of the graphic representation of successes. Conceptual objects constitute the basic units of these codings. (See Table 1.)

To determine the heuristics the subject employed, the coding schema utilized one set of indicators for episodes following a successful construction and another set for the episodes following an unsuccessful construction. The post-success indicators were: (a) procedure for forming the construction, as evidenced by sequence and rhythm of action and verbal protocol; (b) that which is conserved and that which is changed in the attempt to transform one success into another; and (c) any verbalized rationale for why the change in gear-construction would effect no change relative to the goal-criterion. The post-failure indicators were: (a) tests the subject carried out in search of the error (as evidenced by gesticulations, train of visual attention, physical manipulations of the material, and verbal protocol); (b) that which the subject modified in the attempt to transform a failing construction into a success; and (c) the procedure of modification (as evidenced by sequence and rhythm of action, and verbal protocol). (See Table 1 for a much abridged version of the heuristics.)
<table>
<thead>
<tr>
<th>Problem Spaces</th>
<th>Conceptual Objects</th>
<th>Heuristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean</td>
<td>Symmetries (mirror and slide)</td>
<td>1) Effect proper spatial alignment of men-figures</td>
</tr>
<tr>
<td></td>
<td>Straight lines</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Angles (of gear-alignment)</td>
<td>2) Compose an effective shape/fix into an effective shape</td>
</tr>
<tr>
<td></td>
<td>Axes (of gear-alignment)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Shapes (as constituted by gear-elements or gear-configurations)</td>
<td>3) Compose an effective arrangement of discrete elements/fix into an effective arrangement</td>
</tr>
<tr>
<td></td>
<td>Sizes (of gears, or gear-configuration)</td>
<td></td>
</tr>
<tr>
<td>Kinematic</td>
<td>Motions</td>
<td>1) Pre-kinematic pattern:</td>
</tr>
<tr>
<td></td>
<td>Opposite (directions)</td>
<td>-by trial and error and extensive study of motion, construct a set of placements in which the 2 marked gears are each moving head-first</td>
</tr>
<tr>
<td></td>
<td>Alternancy (of rotations)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Series (of gears)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discrete gear placements (defined in topological terms)</td>
<td>2) With alternancy pattern:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-build/correct a set of placements in which 2 marked gears are in kinematically equivalent positions in alternancy pattern</td>
</tr>
<tr>
<td>Dynamic</td>
<td>Agent/patient</td>
<td>Build/modify construction so that both marked gears are pushed in some direction, inferring direction of displacements from the knob-motor</td>
</tr>
<tr>
<td></td>
<td>Pathway of transmission of movement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Inversions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Opposite (direction)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Displacements</td>
<td></td>
</tr>
<tr>
<td>Topological</td>
<td>Contingency (among gears)</td>
<td>1) Pre-arithmetic rule:</td>
</tr>
<tr>
<td></td>
<td>Closure/openness of form</td>
<td>-connect gear-elements into a string or network</td>
</tr>
<tr>
<td></td>
<td>Opposite (direction)</td>
<td>2) Arithmetic rule:</td>
</tr>
<tr>
<td></td>
<td>Parity</td>
<td>-make/correct strings with specific number of intervening elements (Correct rule: odd number of intervening elements)</td>
</tr>
<tr>
<td></td>
<td>String (chain of connected elements)</td>
<td></td>
</tr>
</tbody>
</table>

*Much abridged version of coding schema

Each coder repeatedly views each episode for any of the indicators of a problem space, both indicators of the representation of the task domain and indicators of the gear-construction strategy. The coder also examines the context of prior and subsequent episodes. Although one could argue that this backward and forward viewing biased the categorizing towards a continuity of problem space, this potential effect was compensated by the correlary effect of being able to see subtle changes. This contextual analysis
is needed to capture the meaning of the actions to the subject. Frequently, one episode is part of a more extended action plan and/or course of observation, and thus cannot be validly interpreted in isolation.

There are three possible outcomes for the coding of an episode. If a subset of indices of one problem space are identified, the episode is coded for that problem space. If the indices of more than one problem space are identified, then the episode is coded for both and considered a compound problem space (with one exception, explained below). If the indices of none of the problem spaces are unambiguously manifested, the episode is coded as problem space indetermined ("PS"). The coder records both the outcome, as well as the indices on which she has based her decision.

The exception to this categorization procedure concerns the labeling of the elements. First, in the course of the pilot studies, I noticed that the labels spontaneously given to the materials frequently continued to be used after all other indicators of problem space had been changed. The working hypothesis employed here is that initial spontaneous labeling is a good indicator of problem space. However, the duration of its use is not a good indicator of the duration of the problem space. It appears that labeling may endure as a residual effect of prior spaces. Second, the correct use of the technical term "gears" does not in and of itself imply a dynamic conceptual framework. More generally, we can not infer that a child who has learned a technical label has necessarily mastered the associated conceptual system.

**Coding of the Arithmetic Modifier.** After the problem space has been assigned, the coder considers whether or not the subject quantifies units, as they are defined by this problem space. If such quantification does occur, an "A" is added to the problem space code.

**Resolving Differences in the Two Sets of Codings.** Following the completion of their independent analyses, the coders compare both demarcations of episodes and designation of problem spaces. Disagreements are jointly resolved on the basis of a comparative analysis of the indices used (as recorded by each coder) and reviewing(s) of the tape.

**Role of the NaiveCoder.** A naive coder analyzed a sample of the data, including the first 15 min of four nonrandomly selected tapes and the first 15 min of another four randomly selected tapes. This was to serve as a check on implicit understandings between the trained coders.

**RESULTS**

The mean inter-rater agreement on demarcation of episodes per protocol was 92%. The mean inter-rater agreement on designation of the problem
space(s) per protocol was 90%. Of the episodes, 90.8% were coded to a particular problem space; 9.2% were coded as "PS". The agreement between trained raters and the naive rater for the nonrandomly selected tapes (first 15 min of four tapes) was 95%. The agreement on the randomly selected tapes (first 15 min of four tapes) was 81.3%.

Subjects employed one or more of four different problem spaces: the Euclidean, Kinematic, Dynamic, and Topological. An analysis of the content and pathways of transition revealed that these four could not be subsumed within one expanding problem space.

Subjects in the two age groups began the task in remarkably similar ways. (See trajectories of problem spaces each subject employed, in Figures 2 and 3.) With the exception of the first five episodes of one 8-9-year-olds

![Figure 2. Problem space trajectories: 8-9-year-olds](image-url)
subject, all of the 8–9-year-olds began the task in the Euclidean or Kinematic problem spaces, or some utilization of both (in vacillation or combination). Of the 11–12-year-olds 70% did as well. However, 30% of the 11–12-year-olds began in the Topological problem space, a space no 8- or 9-year-old employed. These differences in initial problem space distributions are not statistically significant.

In contrast to the age groups' similar beginnings, there were marked differences in the problem spaces the two age groups were employing at the end. (See again Figures 2 and 3). Seventy percent of the 11–12-year-olds ended the task in the Topological space, a space that no 8–9-year-old employed under the conditions of this experiment. One 11–12-year-old ended the task operating confidently in the Dynamic problem space, a space no

![Figure 3. Problem space trajectories: 11–12-year-olds](image)

*Arithmetic Modifier severed from Kinematic Problem Space
*Euclidean OS source of variation only

Figure 3. Problem space trajectories: 11–12-year-olds
8–9-year-old was able to sustain. All of the 8–9-year-olds ended the task in the Euclidean or the Kinematic, or some utilization of both, as compared to only 20% of the 11–12-year-olds. These differences in range and distribution of final problem spaces are statistically significant ($\chi^2, p < .05$).

Among the 8–9-year-olds, six subjects exhibited no change of problem space at all. A seventh subject (L, 9.3 F) vacillated between two spaces. An eighth subject (A, 9.5 F) briefly used the initial space in combination with a second. Only two 8–9-year-olds rejected any space they had been using in favor of another. Excluding three 11–12-year-olds who began in the adequate Topological problem space, all but one shifted to a more goal-appropriate problem space.

In the following sections, I describe each of the problem spaces in terms of representation of the task domain, the heuristics that operate upon the representation, and strengths and weaknesses of the space. Then, I analyze whether these problem spaces can be subsumed under one expanding space. I describe the arithmetic modifier, when it is used, and the problems associated with its application. Finally, I consider the form and function of compound problem spaces and vacillating problem spaces.

THE FOUR PROBLEM SPACES

The Euclidean Problem Space

Controlling relative directionality by alignment of the two men

(See Figure 4.) R (9.5, F) begins by placing the two men side by side, with a slide symmetrical matching of figures (a). Then she turns the knob counterclockwise. One man goes head-first; the other, feet-first. "Wait," she says, and turns the knob the other way (b). Following this second failure, she puts the men back into their original positions, side by side, with slide symmetrical matching of orientations. She turns the knob counter-clockwise (c). "Doesn't. No, he's going feet-first." Then she starts to re-align the men again back into the original slide symmetrical, side by side placement state, but abruptly stops. R proceeds with a side by side, mirror symmetrical alignment (d). Again, her construction fails. "Humm." She then changes the positioning of the man going feet-first to below the other gear, and changes the form of symmetrical correspondence of figures, from mirror to slide (e).

Controlling relative directionality through gear-configuration shape

T (12.2, M) See Figure 5.) "I'm going to put the gear in the center of the board this time" (puts (a) in the center of the board). "I'm going to try to make a circle going all around the gear" (puts (b) on either side of the gear). "And have one of the persons at the top right above the gear" (T
Controlling directionality through size of elements and/or their alignments.

C (12.8, F) "Maybe I can put one the same size in the middle, then it'll work." C puts a gear the size of the marked gears in the center of the board, attaches a marked-gear to its right, then the other marked-gear to its left. "Maybe you know if they're all the same size." (C puts the knob in the man-gear on the right and turns.) "Yeah! It works."

Ex: T (12.2, M) "Next try to put the knob at the top and just have a straight row of middle-sized ones down..."
Representation of the Task Domain. The Euclidean problem space is composed of elements and relations of Euclidean geometry, a conceptual system that is actually irrelevant to the control of relative directionality. There are three Euclidean approaches to the control of relative directionality. In the most primitive form, the task domain is represented by two spatial relations: (a) relative placement on the board of the two marked elements, and (b) form of symmetrical correspondence between the orientations of the men figures. The symmetry is one of two types, either mirror or slide. (In slide symmetry, the orientations of the two objects are identical, appearing as if one object had been directly slid into a second position.) Subjects ignore the unmarked gears.

In the other Euclidean forms, subjects attend to unmarked gears as well. In one form, the gear configuration is the most salient entity. Subjects focus on size of gear-elements, and positioning of elements relative to each other and to the perimeter of the board, as these serve to define the gear-configuration shape. In the last form, the discrete gear-element constitutes the most salient entity. The particular gear-configurations are conceptualized in terms of the size of elements employed and/or their alignments relative to each other.

Heuristics. In the most primitive form, subjects assume that proper symmetrical matching of orientation and proper relative positioning are sufficient to make the two men turn the same way. Once a subject has realized that a particular combination of relative placements and symmetry will not work, there are two kinds of possible changes: (a) switch the form of symmetrical correspondence, and/or (b) change the positioning of one man relative to the other. The subject does not consider using unmarked gears as a means to modify the men's relative directionality.

In the form with shape as the most salient entity, that which is constructed, conserved or modified is defined in reference to the shape. In attempting to transform a failure into a success, the subject of this level frequently changes the shape in which the two marked gears are embedded, without changing the connection between them. In the third form, the subject modifies the construction in diverse ways, across which no apparent shape is preserved, or is itself the object of transformation. Instead, the subject tries another arrangement, focusing on size and/or particular alignment of individual gears.

Analysis of Strengths and Weaknesses. The Euclidean problem space is fundamentally inappropriate for the control of relative directionality. Symmetrical correspondence, particular angle of alignments, size of elements, gear-configuration shape, and positioning on the board are each irrelevant to the attainment of the goal criterion. Thus, solutions are not derivable.
from any of these forms. Where success is serendipitously attained, it is attributed to spurious conditions. As would be expected, this problem space has a comparatively low success rate (see Table 2).

A particularly interesting and common type of error is the use of symmetry as a means to achieve correspondence of displacements. Two strategies were based on symmetry. The initial Euclidean form involved symmetrical matching of men-orientations. These subjects vacillate between mirror and slide symmetry, convinced that the correct symmetrical relation between men figures (in addition to the correct positioning of one element relative to the other) should solve the problem. Subjects have perseverated within this ill-fated symmetry strategy from 3 to 13 episodes. (See again excerpt above from R.'s (9.5 F) protocol.)

| TABLE II |
| Percentage of Successful Episodes Within Each Problem Space |
| Age Group | E | K | D | T |
| 8-9 year olds |
| without arithmetic modifier | 28.14 | 48.89 | 25.00 | ___* |
| (231) | (45) | (4) | |
| with arithmetic modifier | 100.00 | 71.43 | 00.00 | ___* |
| (1) | (14) | (0) | |
| 11-12 year olds |
| without arithmetic modifier | 45.80 | 33.33 | 56.67 | 34.78 |
| (65.5) | (40.5) | (15) | (23) |
| with arithmetic modifier | 12.50 | 66.67 | 100.00 | 71.56 |
| (8) | (1.5) | (1) | (54.5) |
| Totals |
| 8-9 & 11-12 |
| with & without | 31.7 | 46.0 | 52.5 | 60.6 |
| (305.5) | (101) | (20) | (77.5) |

*8-9 year olds never used the Topological Problem Space. Numbers in parentheses equal the total number of episodes for that age group x problem space x with/without arithmetic modifier. .5 indicates the problem space was employed in combination with another.

In the second type, the subject builds a bilaterally symmetrical gear-configuration, and places the marked gears in bi-laterally symmetrical positions within the form. For an illustration of this approach, consider a sample of T's (12.2 M) successful constructions, as represented in a page of his drawings. (See Figure 6.)

Similarly, young children seeking to establish equilibrium on the balance beam frequently employ a Euclidean framework. Inhelder and Piaget (1954/1958) ascribe to their youngest subjects on the balance beam task a strategy of symmetrical correspondence in the placing of weights. These
subjects sought a correspondence of distance from the ends of the balance beam, and ignored the more dynamically relevant fulcrum point. It is hypothesized that visual symmetry is one primitive heuristic employed by physics-naive subjects seeking to create identity of actions (as in this task) or equilibrium (as in the balance-beam task).

The Kinematic Problem Space

*Without the alternancy of rotation pattern*

A (9.7, F) (See Figure 7.)
A places a marked gear (a) on the board, and turns it head-first 180 degrees. After attaching another gear (b), she again turns the marked gear briefly feet-first, and then head-first. She proceeds to attach a third gear (see position 1), forming a straight line with the two gears already in place. A then sets the construction in motion; the marked gear goes head-first. A changes the alignment of the third gear on the second gear (see position 2), and again sets them in motion. The man is still going head-first. She then removes this third gear, and places a gear of the same size (d), in the original straight line positioning. Again she sets them in motion. She adds a fourth gear (e), and checks the motions by turning the third gear; the man goes feet-first, then head-first. She adds the other marked gear and puts the knob in the second gear. As she turns the knob, both men go feet-first. “Oops!” she exclaims. She seeks to rectify the movements, not by turning the knob the opposite
direction, but by adjusting the alignment of this last marked gear relative to the fourth. She turns the knob again; both men go head-first. "There!"

With the alternancy of rotation pattern

B (9.11, M) (See Figure 8.) "I'm putting on two more, so that this one (f) will move the same way that this one (e) did. This one (new gear (g) still in hand) will move the same way that that one (d) did, and then I can put this guy (e) on again." After inserting these two gears, B turns the knob slowly, while observing the series-in-motion from one end to the other.
**Representation of the Task Domain.** In the Kinematic problem space, motions are considered to be the pertinent data base. The kinematic subject changes the task domain into a data base of motions, by turning the knob or gear edges before placing one or both marked gears on the board, and/or continues turning after the construction has been evaluated. Motions constitute the basic entity that is represented. Gestures of the hand, or linguistic tagging (e.g., "clockwise" and "counterclockwise") function as modes of representation. Subjects gradually abstract relations between the motions. The first relation they form is that gears next to each other turn different directions. A much later and more powerful relation is the alternancy of rotation across a gear-series.

**Heuristics.** Before abstracting any kinematic relations, the subjects generate or modify a construction, with extensive observations of motion. Subjects transform the observation that two contingent gears move in different directions into the constraint: Separate the two gears that you want to turn in the same direction. One subject (A, 9.7 yrs. F) then formed a more adequate constraint based on this same observation: Place the marked gear next to a gear which is turning the opposite direction to the way you want the marked gear to turn.

With the alternancy pattern, more subjects can formulate a constraint capable of effectively controlling the relative directionality of the marked gears: Build a gear-series with the marked gears in kinematically equivalent positions or, as in the example excerpted above, extend a successful gear-series, maintaining the kinematic correspondence of the two marked men. When a construction fails to meet the goal test, the subject either assumes the alternancy condition has been violated and directly inserts an element, or reevaluates placements in terms of this condition.

**Analysis of Strengths and Weaknesses.** In contrast to the Euclidean problem space, the Kinematic problem space is a fundamentally goal-appropriate framework. The sphere of relative motions of all gears, marked and un-marked, is an effective way of observing one's evolving constructions, of abstracting relations, and formulating constraints for gear-constructions.

The power of the heuristics varies. At first, before the abstraction of any kinematic relation, the heuristics are limited to the weak method of generate and test; or generate a part of the construction, test it, then modify or continue building on. The subject who places the marked gears in kinematically equivalent positions in the alternancy pattern will consistently produce successful constructions. However, this constraint is frequently difficult to carry out. Calculating relative directionality on the basis of this pattern imposes relatively high demands on short-term memory. It is complicated to keep track of how the pattern maps, element by element, onto constructions.
being formed. The difficulty appears to be heightened when the configurations are not restricted to simple chains, or when the motions of the individual are not restricted to simple chains, or when the motions of the individual gears are tagged with a gesture instead of a word.

Another weakness of the problem space is the quality of the understanding it offers. The facts that two gears next to each other turn different directions and that direction of rotation alternates remain arbitrary empirical observations. The fragility of understanding and conviction is manifested in subjects' consideration of exceptions to this pattern. For example, in seeking to predict the relative directionalities in a circle of nine gears, D (D, 12.0 M) postulated, "Somebody's got to move the same way [as the gear next to it]... I guess there has to make an exception somewhere, somewhere in the chain... I think it's these two... wait a second, then you have three... You're going to have three because if it was these two and they all moved the same way... it's four..."

The Dynamic Problem Space

(See Figure 9.) M (11.7, F) reflects "If this (points to both b and c, and turns b slightly counterclockwise) is going to make this one go this way (turns c counterclockwise), which is going to make this one go this way (turns d clockwise), it's going to make the other one (man not yet on board) like that (turns imaginary man-gear clockwise), and if I just use one (e) for this one (f)... they might be going the same way. They are!"

(See Figure 10.) "Because it's like, it's (e) two away (points to b and c) from this (a), so... See, this goes (turns b counterclockwise) it's (c) two away (points to b and d) from the one that has the knob (a), which is providing the direction. And this makes it go (turns b counterclockwise, and points to (a) the opposite direction of this points to (b) And that makes, that makes it go the same direction as this (points to a and d) So this (f) will go the opposite direction of this (a), and so will this (points to b,c,e—finger rests on e), because it's two away from that (a)."
Representation of the Task Domain. A causal linking between the observed motions differentiates the Dynamic from the Kinematic problem space. The Dynamic problem space is based upon the ideas of agents, patients, inversions of directionality, and pathways of transmission of movement. The subject conceptualizes gear-constructions in terms of pathway(s) of transmission of movement, formed by gears acting on other gears, from the point at which the force is introduced.

Heuristics. The goal-criterion is interpreted in terms of the formation or modification of a gear construction, such that the two marked gears are pushed in the same direction. Subjects build two forms of gear constructions: constructions with both marked gears in the same pathway of transmission of movement, and constructions with each marked gear at the end of a different pathway. (See Figure 11, M's record of her successful constructions, for examples of these two forms.)

In either form, the subject can modify a failing construction by the addition of one gear, viewed as a direction-inverter. In the former type, one
effective strategy used to transform one success into another is the insertion of two more gears, because as M argues, adding two "is going to be equal to nothing." In constructions of the second type, the subject tries to build or correct configurations so that the pathways are either identical or functionally equivalent.

Analysis of Strengths and Weaknesses. The Dynamic problem space also imposes substantial demands on short-term memory. This is particularly true in constructions with two or more pathways of transmission of movement. The subject begins by inferring the directionalities of movement in one of the pathways. He or she must remember how the knob was turned in these initial calculations, in order to begin the same way of inferring directionalities in the second pathway. Finally, the subject must also retain the directionality of the first marked gear while inferring the directionality of the second, to be able to evaluate their match. This process appears to be especially demanding when the subject does not use linguistic tags.

Consider K's (8.6 M) difficulties with this approach:

(See Figure 12.) K builds a pathway of transmission of movement, ending with one marked gear: he puts gear a on the board, inserts the knob, and then adds gears b and c. E asks, "What are you doing?" "I'm just trying to like, so when it turns it, this one will move that one." K silently calculates successive displacements, gesturing clockwise, counter-clockwise, clockwise, with a finger. Then he adds a fourth gear (d). "Um, this one (gesturing clockwise above a) makes that one go that way" (gesturing counter-clockwise above b). This one (pointing at c) makes that one go this way (gesturing clockwise above d). And that one (gesturing counterclockwise above d) makes that one go that way (gesturing clockwise above e). K then places the other marked gear, f, next to e; and then adds on the rest of the available gears, without any indication that he is calculating displacements...

K spends 7.5 min drawing the final construction. (See Figure 13). Each time he draws another gear he calculates which way it will turn by inferring the successive displacements from the knob. When the drawing is
completed, he repeatedly calculates the succession of displacements between knob and each marked gear. He ignores the intersection of these paths.

Keeping track of the chains of cause and effect across pathways of transmission of movement poses significant demands on short-term memory. Just calculating the directionalities across the first five gears placed on the board proves to be too much for K. (Note he failed to consider how gear (b) will displace gear (c), and thus gets off in his chain of inferences.) Diverse indications that suggest the STM load exceeds K’s capacity include his errors in calculating across the original pathway, his failure to consider the effect of the other gears outside of this pathway during the construction, and the problems that he has calculating displacements as he draws.

The much more sophisticated and successful M (the F 11.7 subject initially excerpted) limits her constructions to one or two pathways of transmission of movement. However, she also has problems keeping track of directionalities. (See, for example, the error in the excerpt cited. Gears d and g cannot both be going clockwise.)

The strength of this problem space is that it is based on a more adequate understanding of the task domain. The Euclidean problem space offers an irrelevant conceptual framework. The Kinematic can, at best, enable the abstraction of empirical regularities. The Dynamic framework offers an explanation of why the gears next to each other turn different directions, in terms of the effect of the pusher upon the pushed.

The Topological Problem Space

Without the arithmetic modifier

C (12.8, F) In response to E’s question “What are you doing now?” C says “Well, I’m going to connect as many as I can and still make ‘em
DEVELOPMENT OF CHILDREN'S PROBLEM SOLVING

I may just make a big chain here. C continues to add gears one by one, forming a chain around the perimeter of the board. At no point does she count the elements.

With the arithmetic modifier

C (12,3, M) "There's 12 pieces. But two of them have to be men, right? So then, you have to have either three, you have one, three, five, seven, or nine pieces between each one, which makes um, how much? I just said 1,2,3,4,5 ways. Is that right? I think. C proceeds to build the five ways, inserting each time an additional two gears in the intervening gear-string, with the exception of two explicit counter-examples (with two and four intermediate elements). (See figure 14 for C's set of drawings, a set he assumed to be exhaustive.)

---

**Figure 14. C's drawings of his successful constructions**

**Representation of the Task Domain.** The conceptual framework of this problem space is based upon Topological Geometry. Topological Geometry has been referred to as "rubber geometry," because any transformation one could carry out on a rubber form has no effect on its topological properties. The only distinctions are openness or closure of gear-configuration form and connectedness or separation of gear-elements. The Euclidean and Topological are similar in that they are both static geometrical systems.

---

Since the Topological problem space so frequently occurs with the arithmetic modifier (in 70.3% of all Topological episodes) and this combination constitutes the most successful approach to the problem, the Arithmetic-Topological combination is described separately.
However, they are distinct in that the Euclidean distinctions of size, inter-element alignment and shape are not considered to constitute differences in the Topological system.

**Heuristics.** Before the formation of numerical constraints, the subject employs the weak method of generating an element-network, or "string," and testing. Strings are defined here as chains of connected elements, without regard to element size or alignments. The subject who characterizes successful and failing strings numerically soon recognizes the parity rule: Build or modify a construction such that there is an odd number of intervening elements between the marked gears. (See the products of this strategy, as manifested in C's sketches of his successful constructions.)

**Analysis of Strengths and Weaknesses.** The primary strength of the Topological problem space is that it facilitates the efficient enumeration of the complete set of possibilities, as defined by the conceptual system. The most successful subjects in the study were those who applied the arithmetic modifier to the Topological problem space. (See again Table 2.) The approach imposes negligible demands on short-term memory, as subjects need only to count number of intermediary elements. It avoids the difficulties involved in assessing relative motions or displacements, as the space is founded on a static geometry.

The difficulties of the Topological problem space are totally different from those of the prior spaces. The arbitrary quality of understanding constitutes the primary weakness of the problem space. Different protocols have manifested this characteristic in diverse ways. Three forms are described here: confusion concerning which elements to count, the subjects' spontaneous descriptors of the odd/even rule, and poor explanations of relative directionality.

Consider an excerpt from D's (12.0, M) protocol:

(See Figure 15.) "As long as you have an odd number in the middle," states D, as he builds a string with five intermediary elements. "If you
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fake one out, it won't work." D tests his hypothesis, by removing one intermediary element from the string. He then proceeds to construct a long chain of six intermediary elements, and a seventh adjacent to the string. D looks perplexed when the two men turn opposite directions. He removes one intermediary element, and then counts the elements, including in this count the irrelevant element adjacent to the string: "I think I put the wrong number together." He sets the construction into motion, and looks perplexed that it works. He repeats his rule: "All you have to do is get an odd number in the middle and you have it."

With the discovery of the odd/even regularity, no subjects thought they had understood the materials. The odd number condition has been variously referred to as the "trick," the "secret," a "co-incidence" and the way the Experimenter "fixed the game."

C (12.0 M) was the most confident, and most directly successful Topological subject. After less than three min on task, C claimed he was "on the roll"; that there were some five, or six and seven ways, however many constructions you could make, with 12 gears, maintaining an odd number of intermediary elements. Following his quick enumeration of the set (See again Figure 14), he was shown the board with the two marked gears touching and asked to predict and explain what would happen when he turned the knob.

C looks surprised. "Oh! There are six ways!" (Long pause) "No. When it turns, it'll go the opposite way. Because when one gear is next to another, it turns, it's made to go clockwise and then counter-clockwise, to something like the balance, or to make in a pendulum. It started out in a clock or something. I'm not smart about (Pause). I don't know. That's the way it goes, because the two gears are together and when one's moved, the other's moved the same way, from the same position. I know why, but I don't know the real reason why. When it is turned, it pushes the other one down, in a way. Like, in a way. Heah, it doesn't! It lets the other one go the opposite way. It pushes it down, while it (the next gear) it's going up. I think that will (pause). I just know every odd number (pause) would be (pause) I'm probably wrong about this whole thing. Every (pause) every odd number of the gears makes the odd numbers turn the same way, and the even numbers turn the same way. Was I right? Will you tell me?

The surface-level understanding revealed by this most efficient and most accurate of Topological subjects (in fact, by far the most efficient and accurate of all subjects) contrasts sharply with the 11-12-year-old Dynamic subject's response to the same post-procedure question. After a brief confusion concerning mirror and slide symmetry, M (11.7 F) argues that these two contingent gears would turn different ways, unless they pivoted around each other. In other words, she re-invented polar gears.

The Topological is an elegant approach, in that no features unnecessary for the formation of successful constructions are considered. However,
the models which are exclusively Topological offer no model of movements for the mechanism. The topological odd/even rule is simply a regularity, abstracted from the gear-construction in a state of no-motion.

The Arithmetic Modifier

*Forms of the Arithmetic Modifier.* The arithmetic modifier involves a numerical characterization of the units of meaning, as defined by the semantics of the problem space to which it is applied. Examples of the arithmetic modifier with the Topological problem space were described in the previous section. The arithmetic modifier applied to the Euclidean problem space involves a counting of circle-elements. For example, C (12.3 M) states, "I'm going to make them do head-first somersaults with four circles." When this fails he tries another line of four, with different size gears and a different orientation on the board surface.

Typically, the arithmetic modifier applied to the Kinematic problem space is derived from the alternancy pattern. Since every other gear ("spin-piece," "spinner" or "wheel," in the kinematic subject's venacular) turns the same way, inserting two more gears into a successful gear-series in another success. R comments, "I'm putting on two more to make another round. Then it'll go around again."

The only example of the arithmetic modifier applied to the Dynamic problem space is based on the effect of multiple inversions. M (11.7 F) reasons that one intervening element will be functionally equivalent to three. Her confidence level is so high, she directly records the three intervening elements as a solution, without first building the construction.

*Utilization of the Arithmetic Modifier.* The older subjects were much more likely to use the arithmetic modifier. (See Table 3.) However, most of this cross-age difference can be accounted for by a combination of two other factors. First, there was a strong tendency to use the arithmetic modifier

<table>
<thead>
<tr>
<th>TABLE III</th>
<th>Percentage Employment of Arithmetic Modifier</th>
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<tbody>
<tr>
<td>Age Group</td>
<td>Problem Space</td>
</tr>
<tr>
<td>8-9-year-olds</td>
<td>K</td>
</tr>
<tr>
<td></td>
<td>D</td>
</tr>
<tr>
<td>11-12-year-olds</td>
<td>K</td>
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<td>D</td>
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</table>

*8-9 year olds never used the Topological Problem Space.
Note: Numbers in parentheses equal the total number of episodes for that age group and problem space.
with the Topological problem space. Second, no 8- or 9-year-old ever em-
ployed the Topological problem space.

Liabilities of the Arithmetic Modifier. The possible benefits of such formal-
ism are comparatively obvious; for example, greater efficiency of task reso-
lution or ease in identifying patterns. The more intriguing question is the
possible problems arising from arithmetic formalizations.

The primary liability identified in this data set is the dissociation of
the arithmetic from the semantic referent. Consider, for example, J's (12.2,
M) protocol. J abstracted the idea of odd versus even as the key to success-
ful constructions, but lost both the implications of odd versus even, as well
as the constraints that odd/even exercised on the formation of successful
constructions.

First, J observed that gears in a series alternated directionality of rota-
tion. Soon thereafter, he realized that if he had a series with the two marked
gears at each end going the same direction, then he could make another suc-
cessful construction by inserting two gear-elements into the series. There
quickly followed the idea of directional correspondence of the marked gears
with an odd number of intervening gears.

Although he continued to maintain "There is no other way to think
about it, by odds and evens," he lost track of the semantics to which odd/
evén applied. After building a circle that jammed, he concluded that even
made things jam, and odd let things turn, or perhaps it was the other way
around. He constructed a long series of multiply-jammed gear-construc-
tions, assuming that it should unjam if he had an odd number on the board
(regardless of their arrangement); or if an odd number failed, he assumed
he needed an even number (again, in any arrangement).

Another less extreme example of the confusion is manifested in D's
(12.0 M) protocol excerpted above. Recall that D was convinced it should
work with odd, but didn't know what elements to consider in the odd/even
calculation. D counted one element adjoining the gear-string connecting the
two marked gears.

The literature suggests that physics-naive students tend to approach
physics problems by means of the relatively quick application of a formula.
They do less preliminary qualitative reasoning and model building than ex-
erts (cf. Larkin, 1977). It is hypothesized that this difference in strategies
stems in part from this kind of weak mapping between the semantics of the
task domain and the symbolic and/or numerical characterization thereof.

Compound Problem Spaces

In 8 of the 20 protocols, indicators of more than one problem space were
manifested in the same episode. (See Figures 2 & 3). Examination of these
occurrences revealed that the different problem spaces were serving discrete functions. The two primary forms of combination were (a) rule of constructing from one problem space, and source of variation from a second, and (b) framework for conceptualizing the configuration from one problem space, and framework for observing the results from another.

**Rule of constructing from one problem space, source of variation from a second**

(12.3 M) "I think it only works with odd number of things in the middle." C proceeds to construct six of the next seven configurations, with an odd number of intervening elements, changing particular positions on the board ("Using nine pieces again, but this time the men aren't going to be in the same place."). or size ("Now I'm just going to use the little and bigs"). In the one exception, he builds a chain with eight intermediary elements. When this fails, he immediately counts the elements: "Ten! No wonder it didn't work."

Subjects used one form of this combination. The Topological functioned as rule-source, and the Euclidean as source of variation. If one utilizes only the Topological framework, the number of goal-states is severely limited. The subject, challenged to find all of the solutions, may consider the resulting set woefully inadequate. As J (11.11, F) commented, "It's just every way you do it always seems the same. It's like different pieces but it's the same idea... the odd number."

If one admits Euclidean changes into the conceptualization of goal-states, the number of variants expands dramatically. Changes in size of elements as well as changes in gear alignments are then considered new goal-states.

**Framework for conceptualizing the construction, framework for observing the results from a second**

A (9.7 F) slowly builds a bi-laterally symmetrical form, frequently checking the motions that result (See Figure 16):

A picks up a gear (a) and places it in the center of the board. She adds on a second gear (b), and turns it 180°. "I'm experimenting." She attaches a third gear (c) to the gear (b), and turns it 90°. She adds on a marked gear (d) on turns it 180°. She puts the knob in the marked gear,

![Figure 16. A's gear construction](image)
and turns it 90° in each direction, causing the man to go feet-first and then head-first. She simultaneously picks up two more gears (e and j) and places them in positions that complete a bi-laterally symmetrical form.

The two problem spaces that appeared most frequently in combination (in 5 of the 20 protocols) were the Euclidean and the Kinematic. Examinations of these episodes indicate that the Euclidean typically serves as a framework for conceptualizing possible construction alternatives. The Kinematic functions well as a goal-appropriate means for observing the results. Prior to the abstraction of relations, the Kinematic problem space offers no clear means for conceptualizing the gear-construction one is building.

**Vacillating Problem Spaces**

Vacillation is defined here as the return to a prior problem space at least two times. The problem space can be either a single or compound space. This form of change must be clearly distinguished from those cases in which a subject comes to reject one space as inadequate, and shifts to the stable utilization of another.

In 2 of the 20 protocols (see Figures 2 & 3, subjects L (9.3F) and J (12.2 M), the subject vacillated between problem spaces. In each of these cases, vacillation occurred between two particular problem spaces, the Euclidean and the Kinematic, singly and in combination. The level of the Kinematic involved in these vacillations was the more primitive, the attention to motion prior to the abstraction of empirical regularities adequate to constrain future constructions.

The adequacy of the different problem spaces, as well as their different levels, varies according to specific function. It is hypothesized that the primary catalyst for problem spaces in combination or in vacillation is the same: the perceived need for a function that cannot be adequately fulfilled by one space alone, as it is currently understood.

**One Expanding Problem Space or Many?**

Whether the four problem spaces applied to this task can be viewed as the expansion of a single space needs to be considered in light of the particular pathways of transition across the spaces. The three most common pathways involved transitions from (a) Euclidean to Kinematic, (b) Kinematic to Euclidean, and (c) Euclidean to Topological. To this list I add a fourth, Kinematic to Dynamic. Although this pathway was traversed by only one subject, it constitutes the only route observed in this data set to the Dynamic.
Only one of these four pathways, the shift from Kinematic to Dynamic, can be viewed as a simple problem space extension. The Kinematic is based on motions; the Dynamic, on motions as well as the pathways of transmission of movement by which these motions are created and explained.

Each of the other transitions entails either substitution or some partial incorporation. The shift between Euclidean and Kinematic involves a fundamental shift in that which is encoded, and in the data base where the subject searches for invariances. Kinematic subjects define the placements that effect these motions in topological terms (e.g., gears "next" to each other turn different directions).

In the shift from Euclidean to Topological, the subject comes to recognize that Euclidean features with arithmetic constraints are sufficient, but not necessary, for goal-attainment. The Euclidean features become replaced by the more parsimonious topological feature. A common carry-over in this transition is the arithmetic modifier.

No single expanding space emerges from the four problem spaces, as they are applied in this context. Instead, the four constitute a complex model of partial or complete incorporation, and substitution. However, the possibility remains that the genetic order of problem space formation would more closely approximate one expanding space.

**DISCUSSION**

What are the implications of this study for a fundamental issue in Cognitive Development: "What develops?" (Flavell, 1963; Siegler, 1978); that is, what cross-age differences might be responsible for the differing performance of 8-9- and 11-12-year-olds, in terms of range and distribution of problem spaces employed and frequency of problem space shift?

Why did no 8-9-year-old subject employ the Topological Problem Space? Why was adaptive shift so infrequent among 8-9-year-olds, and so frequent among 11-12-year-olds? What differences in the minds of 8-9-year-olds and 11-12-year-olds could underlie these differences in performance? I consider four possible factors, attributing to these effects: (a) different problem space repertoires, (b) adequacy of problem space indexing, (c) adequacy of understanding the implications of an empirical event, and (d) meta level of the means. I also consider the role the particular task could have played in these findings.

**Repertoire of Available Problem Space Hypothesis**

Can the cross-age differences found in range and shifting of problem space be explained in terms of difference in problem space repertoire? Do 8-9-year-olds fail to apply the Topological framework because they do not "have" it?
Piaget and his colleagues have presented a series of studies (cf. Piaget & Inhelder, 1948/1967) indicating that Topological Geometry constitutes the first system for representing space, followed by Euclidean and finally Projective Geometry. According to this body of work, by age 7 the child would have the Topological framework. These conclusions offer no explanation for this later application of the Topological, and no support for the hypothesis that the Topological is not within the 8–9-year-olds repertoire of problem spaces.

These conclusions have been modified by Laurendeau and Pinard’s (1970) masterful reconsideration of the development of spatial concepts. They report that between 3.6 and 5.8 years of age “topological considerations are the only ones operating” (p. 434). However, they qualify these conclusions in a manner of direct relevance to this discussion.

One qualification concerns the level at which the child understands the Topological system. Laurendeau and Pinard argue, “these first topological concepts in the very young child still lack any abstract formalization or specification; they remain implicit and poorly defined” (p. 427). The initial fuzzy relation between Euclidean and Topological features constitutes one manifestation of this poor definition.

Laurendeau and Pinard’s “topological” subjects sometimes made distinctions involving Euclidean properties, by means of Topological schemes. For example, with their finger tracing the perimeters, some subjects could differentiate curvilinear from rectilinear forms. The authors posit that the tracing of rectilinear forms suggests interruption, which in turn suggests the Topological distinction of discontinuity.

The Topological problem space identified in this study differs radically from the topological thinking of the young child, as described by Laurendeau and Pinard. When subjects shift from the Euclidean to the Topological Problem Space, they do so with explicit recognition of the attributes they no longer consider important:

For example, C (12.1 F) begins in the Euclidean problem space: “...I can put one the same size in the middle, then it'll work. Maybe if they're all the same size...” Later she rejects size as a goal-relevant feature: “If you put three in the middle... I don’t think it matters what size it is.”

In summary, Laurendeau and Pinard have found that the preschool child uses Topological Geometry. However, the preschooler’s topological geometry is not an explicit well-formalized system, clearly differentiated from the Euclidean. The current literature does not indicate whether or not at age 8–9 the topological framework would have acquired these system features.

These findings suggest an inadequacy of the original hypothesis. It is insufficient to consider whether or not a child “has” a given problem space. We need to also consider the level of formalization and the clarity of differentiation from other related systems.
Adequacy of Problem Space Indexing

Another possibility is that the 8-9-year-olds have the Topological problem space, at a functional level of formalization, and yet do not evoke the space under these task conditions. A second answer to the question "What develops?" concerns the adequacy of the particular problem space evoked for the task domain and goal.

In information processing terms, this capacity could be attributed to a superior indexing of problem spaces. Indexing could be superior in two senses. Either indexing is more constrained, thus avoiding instances of the over-generalized use of the space. Or, there is more complete indexing of problem spaces to appropriate spheres of application, thus avoiding instances in which the problem space is not associated with the task domain to which it could be validly applied. This later form of inadequate indexing could be a factor in the 8-9-year-olds' not evoking the Topological framework in this context.

Piagetian theory (cf. Piaget, 1975/1977) would view this problem of evoking an appropriate problem space in terms of the core assimilation-accommodation model. What is the adequacy of the "field of application" of the problem space; that is, the scope of contexts to which it is applied? Second, what is the "accommodation standard" of the problem space; that is, the degree to which the thinker can adjust the problem space to accord better with the task domain, short of rejecting or restructuring the space?

The expert/novice literature reveals distinctions in the facility to evoke appropriate knowledge. Experts have been found to have quick recall of relevant knowledge (Ericsson & Chase, 1982). Novices frequently do not even realize that they know the relevant information (Allwood & Montgomery, 1981, 1982).

In this data set we saw that 11-12-year-olds were more likely to evoke, at least eventually, an appropriate problem space. The indexing mechanism involved remains elusive. An intriguing pair of questions arises here. What is the nature of the element that evokes the more appropriate problem space? Does attending to another attribute (e.g., size of gear) evoke a different systemic approach? Or, does thinking about the problem in a new way lead to the attending to a different set of attributes?

Newell and Simon (1972) argue that "the closed character of the problem space is closely akin to functional fixity. The subject immerses himself in an informational environment that evokes only elements belonging to that environment" (p. 819). What is the unit, systemic or elemental, by which such functional fixity is disrupted? Is it, to quote Duncker (1945), a "suggestion from below" or a suggestion "from above"?

The effect of the indexing, in terms of knowledge activated, is comparatively easily inferred. The unit of the pathway(s) by which the knowl-
edge is evoked, and developmentally, how it becomes established, is much more elusive.

**Adequacy of Understanding the Implications of an Empirical Event**

Detailed analysis of the tapes suggest that the two age groups interpret the implications of an empirical event differently. Two cross-age differences concern (a) the differentiation of sufficient from necessary conditions, and (b) the form of theory testing. Each of these distinctions has important implications for the rejection of an inappropriate problem space.

**Differentiating Sufficient from Necessary Conditions.** The shift from the Euclidean to Topological frameworks exemplifies the understanding that sufficiency does not imply necessity. Protocols as well as task analyses reveal that the advance from the Euclidean to Topological problem space entails an understanding that multiple conditions (e.g., numerical, element-size, inter-element alignments, shapes) regularly resulting in success to not imply this full set of conditions is necessary. There is one condition, contingency, numerically qualified, that can parsimoniously replace any or all of these Euclidean constraints.

Four of the eight 11–12-year-olds who did not begin in the Topological space eventually made this particular shift at some point in their protocols; none of the 8–9-year-olds made this shift. The 11–12-year-olds appear more ready to question that conditions regularly leading to success may not be necessary conditions for that success.

A study directly related to this change is Siegler’s (1976) investigation of the sufficient and/or necessary conditions under which children attribute causality. In Siegler’s study, 5- and 8-year-olds sought to figure out what made a light bulb come on. Was it the insertion of a card into the “card programmer” or a “computer”? He found that the 5-year-olds attributed causality under conditions of necessity alone \((b - a)\) or sufficiency alone \((a - b)\), or both necessity and sufficiency \((a - b)\). The 8-year-olds attributed causality under only conditions of both necessity and sufficiency. The developmental shift found in the Siegler study accords with the current study. I hypothesize that the differences between the two studies in the ages when the shift occurs can be attributed to differences in task difficulty.

**What Constitutes an Effective Test of a Theory?** The means by which the different age subjects sought to test their theories provides another context of implication. The younger subjects appeared to assume that a new positive instance of their theory implied its validity. Those who had explicit hypotheses, and who sought to test them empirically, did so by means of additional positive instances. For example, L (8.9, F) tested her theory, “If a big one is
on the right, at least one of these (the marked men) goes backwards," by placing a big one again on the right. No 8- or 9-year-old was observed testing a theory by means of negative predictions. Seventy percent of the older children tested their theories by means of negative predictions at least once.

These findings accord with Karmiloff-Smith and Inhelder's (1974/1975) observations of subjects, 4.6 to 9.5 years of age, trying to balance various blocks on a narrow beam. The older subjects gradually came to recognize the outcome of some events as counter-examples to their implicit theories (or "theories-in-action"). They did not, however, intentionally formulate counter-examples as theory-tests.

**Meta Level of the Means**

The 11-12-year-old subjects more frequently stop searching for more solutions, in order to study the means. This involves a shift from a success focus to an understanding focus: The procedures that before functioned as goal-appropriate or goal-inappropriate means are transformed into objects of investigation.

For example, J (11.11 F) knew the odd/even rule well. She first stated the rule after a construction with two elements between the marked gears failed. Following a test of the negative predictions of her theory (with four intermediary elements) and five constructions built according to the rule, she again considered the case of two intermediary elements. This time, the goal-inappropriate procedure or prototypical failing construction became an object of investigation, and she figured out why the procedure had failed:

"I want to see if, how, you put two in between them, how the ones in the middle turned, to make the other (marked gear) go the wrong way."

After setting up the configuration (See Figure 17), she turns it slowly, while closely observing the motions. Still turning and observing, J exclaims. "Oh, I see! This one (b) goes the opposite way, to get this (c) to turn a frontal somersault. This one (a) . . . goes the opposite way of this one (b), which is making this (c) go the right way. So it (b) turns this one (d) the wrong way."

![Figure 17. J's gear construction](image)
In this data set, the shift from goal-seeking to understanding frequently directly preceded a shift in problem space. The younger subjects' protocols manifested little of this kind of subgoalining. I hypothesize that the absence of such shifts, from means as tools, to means as objects of investigation, constitutes one factor in the younger subjects' perseveration in an inappropriate problem space.

The Particular Task as Factor

To what extent are these results generalizable? Conversely, to what extent are they a product of the domain and particular task? I consider this question in two sections, as it pertains to two components of the results: (a) the problem spaces employed, and (b) the frequency of problem space shifting.

Generalizability of Problem Spaces Employed. I cannot infer that the 8–9-year-old subjects would not have employed the topological or dynamic problem spaces in response to other tasks, or even under minor variations of the one presented. The research literature indicates that even small differences in the task instructions can affect how the subjects approach the problem. For example, Duncker (1945) found that expressing the instructions in the passive voice led to the investigation of a totally different set of possibilities than the instructions in the active voice.

On the basis of this data set, I can conclude that 8–9-year-olds have, in some form, the Euclidean and Kinematic problem spaces. Under conditions of this study, I can only conclude that 8–9-year-olds do not evoke the Topological, and seldom evoke the Dynamic.

Generalizability of Problem Space Shifting. An examination of the literature in adult and children's problem solving reveals it is surprising that any of the subjects shifted problem space. There is a strong tendency for problem solvers to conserve problem space. According to Simon and Hayes (1979):

Inventing a new representation or shifting from one problem representation to a more appropriate one are acts of human creativity that occur rather rarely, even with intelligent persons. (p. 473)

The first issue to consider here concerns why so many subjects did shift. A number of atypical characteristics of this particular task could contribute to this effect. First, the time spent on task is longer than usual for this literature, and allows more learning to take place. Second, the request to find all the solutions necessitates going back over the task repeatedly. Another study which also asked for multiple solutions (Anzai and Simon's, 1979 investigation of a single subject working on the Tower of Hanoi puzzle) also
found this kind of qualitative shift in task representation. Third, the experimental procedure provides immediate feedback to the subject. Finally, the first problem space most subjects use is inappropriate. Presumably, an inappropriate space combined with conditions of feedback would have a facilitative effect on problem space shifting.

The shifting pattern found in this research leads to both methodological and substantive conclusions. Methodologically, this particular set of task conditions appears well-suited to the investigation of representational or problem space shifts.

Substantively, given these conditions seemingly conducive to such shifts, and given this particular task, the 8–9-year-olds tend not to shift; many of the 11–12-year-olds do shift. We cannot assume that under conditions of more time on task and/or simpler task conditions that the younger children might not shift as well. The fact that the 11–12-year-olds shifted more often than the 8–9-year-olds does not warrant the conclusion that the ability to shift adaptively constitutes a dimension of development. However, micro-analysis of the protocols does suggest that such a change is taking place.

CONCLUSIONS

On this particular task, involving the control of relative directionality among gears, subjects employed four different problem spaces. In the Euclidean, subjects represented the task domain and conceptualized the production strategies in terms of (a) relative orientations of the men figures on the two marked gears, (b) shapes formed by the gear-configurations, or (c) size and/or alignment of the gear-elements. In the Topological, subjects' thinking on the problem was based on distinctions of openness or closure of gear-configuration form, and connectedness or separation of discrete gear-elements. Topological subjects correctly assumed that element size and inter-gear alignment were irrelevant to goal-attainment. In the Kinematic, subjects considered motions to be the pertinent data base, and gradually developed production strategies utilizing the kinematic relations they abstracted. In the Dynamic, subjects sought to predict and explain motions, in terms of agents, patients, inversions, and pathways of transmission of movement. The arithmetic modifier could be applied to any of these problem spaces. Its application resulted in the numerical characterization of gear constructions and/or production strategies.

The two age groups began the task in remarkably similar ways. Ninety percent of the 8–9-year-olds and 70% of the 11–12-year-olds started in the Euclidean and/or Kinematic spaces. Conversely, the age groups finished the session in largely nonoverlapping problem spaces. All of the 8–9-year-olds
were using the Euclidean and/or Kinematic spaces. Seventy percent of the 11-12-year-olds were using the Topological.

Two primary questions arise from these results. Why is it that no 8-9 and yet most 11-12-year-olds employ the efficient Topological problem space? Why is adaptive shifting so unusual among the 8-9-year-olds and so frequent among the 11-12-year-olds? On the basis of a microanalysis of the protocols and a review of related literature, I propose four factors.

First, although the literature indicates that the 8-9-year-olds have utilized an implicit and fuzzy form of Topological as representational framework even as preschoolers, we have no evidence that they now understand Topology as an explicit system, clearly differentiated from the Euclidean. The decision to reject Euclidean features in favor of the Topological requires this kind of well-defined formalization.

Second, the literature has documented that having some knowledge does not necessarily imply knowing when to use it. The 8-9-year-olds may well have a more adequate framework, and yet not have the indexing that leads to its evocation in this particular situation.

Third, both the literature and microanalysis of these protocols suggest older children have a better understanding of the implications of an empirical event. The 8-9-year-olds in this study did not differentiate sufficient from necessary conditions, and hence had no reason to look beyond a successful set of Euclidean constraints. The 8-9-year-olds also considered another positive instance of their theory as implying its validity, and did not test its negative predictions. Most (70%) of the 11-12-year-olds tested their theories by means of negative predictions at least once.

Fourth, the 11-12-year-olds demonstrated a greater propensity to step back from the process of goal-attainment, and take some successful or unsuccessful means as an object of investigation. This kind of change in focus often directly preceded a shift in problem space.

The particular kind of task used in the study constitutes a fifth factor. Problem space shifting is the exception in the research literature. A number of characteristics of the task could have contributed to the relatively high rate of shifting found here: the comparatively long experimental session, the request to find all the solutions, immediate feedback, the numerous and radically different ways the task could be approached, and the fact that the approach most frequently used first happens to be irrelevant. I argue that this genre of task is well suited to the investigation of problem spaces and problem space shifting.

Finally, analysis of the pathways of transition across the four problem spaces applied to this task revealed a complex picture of partial or complete incorporation and substitution. These spaces cannot be viewed as the expansion of one space. These findings contrast with the model of knowledge children apply to solve physics problems, in terms of gradually more complete encoding, allowing more complete processing.
Adaptive shifts have powerful implications for the success and efficiency of the problem-solver, as well as the quality of his or her understanding. We need more adequate knowledge of the conditions under which adaptive shifts do occur, and the mechanism of their transition.

REFERENCES


