Interpretation of Scientific or Mathematical Concepts: Cognitive Issues and Instructional Implications

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Scientific and mathematical concepts are significantly different from everyday concepts and are notoriously difficult to learn. It is shown that particular instances of such concepts can be identified or generated by different possible modes of concept interpretation. Some of these modes use formally explicit knowledge and thought processes; others rely on less formal case-based knowledge and more automatic recognition processes. The various modes differ in attainable precision, likely errors, and ease of use. A combination of such modes can be used to formulate an "ideal" model for interpreting scientific concepts both reliably and efficiently. Comparisons are made with the actual concept interpretations of expert scientists and novice students. The discussion elucidates some cognitive and metacognitive reasons why the learning of scientific or mathematical concepts is difficult. It also suggests instructional guidelines for teaching such concepts more effectively.

1. INTRODUCTION

1.1. Interest of Cognitive Studies of Scientific Concepts

There are several reasons why it is interesting and practically important to achieve an improved understanding of the thought processes involved in using and learning scientific concepts (including mathematical concepts).

Cognitive Interest. The interpretation of scientific concepts involves significant complexities transcending those of everyday concepts. Hence, it is...
particularly interesting to understand the underlying cognitive processes and to elucidate scientists' tacit knowledge.

**Importance for Scientific Problem Solving.** The ability to interpret and use numerous special concepts (such as "acceleration," "electric field," "derivative," . . .) is an essential prerequisite for solving problems in science or mathematics. Indeed, many students' major problem-solving difficulties can be traced to deficiencies in their abilities to interpret the needed basic concepts. These deficiencies constitute a formidable impediment in attempts to teach students scientific or mathematical problem-solving skills (Reif, 1983; Reif & Heller, 1982; Schoenfeld, 1985).

**Educational Importance.** Courses in mathematics, physics, chemistry, and other quantitative sciences devote a major fraction of their time teaching the basic concepts of these disciplines. Many students experience substantial difficulties in learning these concepts. Furthermore, common teaching methods are fairly ineffective. Indeed, recent investigations provide mounting evidence that many students, emerging from such science courses, acquire merely a nominal knowledge of the scientific concepts ostensibly learned by them, are unable to interpret and apply these concepts flexibly, and often exhibit prescientific conceptions or gross scientific misconceptions (Caramazza, McCloskey, & Green, 1981; Clement, 1982; Cohen, Fylon, & Ganiel, 1983; Halloun & Hestenes, 1985a, 1985b; McCloskey, Caramazza, & Green, 1980; McDermott, 1984; Trowbridge & McDermott, 1980, 1981; Viennot, 1979).

**Limitations of Past Work.** The last several years have seen many valuable investigations of students' naive conceptions, preconceptions, and misconceptions in scientific domains such as mechanics, electricity, heat, and others. These investigations include those cited in the preceding paragraph, reported in some books (Driver, Guesne, & Tiberghien, 1985; Lesh & Landau, 1984; West & Pines, 1985), and discussed at several international conferences (Centre National de la Recherche Scientifique, 1984; Helm & Novak, 1983). However, with a few exceptions (e.g., diSessa, 1983; Reif, 1985), most of this work is descriptive rather than analytic, nor does it yield much specific guidance about how to teach scientific concepts more effectively.

**1.2. Scope of Present Work**
To transcend some of the limitations of past investigations, the present paper focuses less on what particular conceptions or misconceptions are exhibited by students or experts, and more on how scientific concepts are interpreted. Thus, the aim is to understand better the underlying knowledge and thought processes involved in interpreting and using scientific concepts...
and then to exploit the resulting insights to teach scientific concepts more effectively.

The following pages are concerned with basic scientific or mathematical concepts of the kind commonly encountered in high school or college courses. Such concepts are of two kinds. Some are "entity concepts," denoting either particular entities (like "the sun") or generic entities (like "triangle," i.e., any member of the class of three-sided polygons.) Others, more complex and particularly important in science, are "property concepts" which describe other concepts (and are thus functions of these other concepts considered as independent variables). For example, "area" is a property concept describing the independent variable "surface" by associating a specified number with every surface. Similarly, "acceleration" is a property concept that associates a specified vector with three independent variables, namely, a "particle," a "reference frame" used to specify position, and a "time."

The present investigation focuses on the "basic interpretation" of these concepts, that is, on the following relatively simple but important problem-solving tasks: (1) "Identification tasks" requiring one to identify whether a specified entity or value represents properly a particular instance of a general concept (e.g., to identify whether a specified vector represents the correct value of a particle's acceleration in a particular situation). (2) "Construction tasks" requiring one to generate a particular instance of a general concept (e.g., to find the value of a particle's acceleration in a particular situation).

There are several reasons for limiting the scope of the present discussion to such basic interpretation tasks: These tasks are appreciably more complex than one might naively suspect. Deficiencies in such basic interpretation tasks account for many of students' difficulties in science courses. Finally, the ability to perform basic concept interpretations is an essential prerequisite for scientific problem solving and for richer kinds of "concept understanding."

1.3. Central Questions
The interpretation of a scientific concept involves the retrieval from memory of some stored knowledge about the concept, and an interpretation process used to apply this knowledge in a specific situation of interest. A particular concept may thus be interpreted in various possible ways, depending on what stored knowledge is retrieved and how it is processed. Hence, there arise the following central questions:

- What are some different possible ways (or "modes") of interpreting a scientific concept?
- What are some of the advantages and disadvantages, including likely errors, of each such mode of concept interpretation?
- What would be the characteristics of an "ideal" mode of concept interpretation that is both reliably effective and cognitively efficient?
What concept interpretation modes are commonly used by expert scientists or novice students?
What are some of the implications for the learning and teaching of scientific concepts?

The following pages attempt to provide partial answers to the preceding questions. In particular, Sections 2 and 3 discuss various formal and non-formal modes of concept interpretation, point out some significant differences between them, indicate the major advantages and disadvantages of each, and give some illustrative examples (taken from various observations, interviews, and some detailed investigations of subjects interpreting the physics concept "acceleration"). Section 4 attempts to specify how scientific concepts can be interpreted both effectively and efficiently, and then makes some comparisons with observed concept interpretations of expert scientists and novice students. Finally, Section 5 comments briefly on some implications for the learning and teaching of scientific concepts.

2. FORMAL CONCEPT INTERPRETATIONS

The different modes of concept interpretation discussed in this section are "formal" in the sense of being deliberately designed to achieve the unambiguity, precision, and generality needed for scientific purposes. They contrast with the less formal interpretation modes discussed in the next section.

2.1. Declarative Specification

This mode of concept interpretation relies on stored declarative knowledge defining a concept by an explicit specification of its characterizing features. The process of interpreting the concept in any particular instance then involves the following major steps: (1) Invoking the stored declarative specification knowledge; (2) retrieving or elaborating its detailed description; (3) devising a procedure for using this knowledge to identify or construct the concept; (4) implementing this procedure in the particular instance of interest.

The following examples, illustrating declarative specifications of several concepts, will again be used later to illustrate constraining interpretation modes for the same concepts.

Example 1: Resistors in parallel. Two resistors are said to be connected "in parallel" if one terminal of the first is directly connected to one terminal of the second, and if the other terminal of the first is directly connected to the other terminal of the second. (According to this concept specification, the resistors \( R_1 \) and \( R_2 \) in Figures 1a and 1b are connected in parallel, but those in Figure 1c are not because of the presence of the intervening batteries.)

Example 2: Angle between two vectors. The "angle between two vectors" is the angle (between 0° and 180°) enclosed by the arrows that represent these vectors.
and emanate from the same point. (Accordingly, the angle between the vectors \( \mathbf{A} \) and \( \mathbf{B} \) in Figure 2a is 120°, as indicated by the arrows drawn in Figure 2b.)

**Example 3: Component vector.** The "component vector of a vector \( \mathbf{A} \) along a direction \( \mathbf{i} \)" is that vector, parallel to \( \mathbf{i} \), which yields the original vector when it is added to another vector perpendicular to \( \mathbf{i} \). (According to this specification, the component vectors of \( \mathbf{A} \) in Figures 3a or 3b are, respectively, the vectors \( \mathbf{A} \), indicated in Figures 3c or 3d.)

**Example 4: Acceleration.** The acceleration of a particle is "the rate of change in its velocity with time." This specification can be expressed more compactly by the formula \( \mathbf{a} = \frac{\text{dv}}{\text{dt}} \), where \( \mathbf{a} \) denotes the acceleration of the particle, \( \mathbf{v} \) its velocity, and \( \mathbf{t} \) the time. (The bold-faced letters denote quantities which are vectors, i.e., which are characterized jointly by a magnitude and a direction.)
A declarative concept specification has the following advantages: (1) It can be stated in compact form and is thus also easily remembered; (2) It identifies explicitly some of the most essential features characterizing a concept; and (3) It can be made precise and general.

A major disadvantage of a declarative specification is that the actual process of interpreting it may be difficult. Indeed, considerable problem solving may be needed to translate the declarative specification into a procedure for identifying or constructing particular instances of the concept. For example, the process is quite simple for "resistors in parallel" (Example 1), and only somewhat more difficult for the "angle between two vectors" (Example 2). But it is appreciably more difficult for the "component vector." Thus, quite a few students have difficulties using this specification to find the component vector of A in the configuration of Figure 2b. Finally, as we shall see, the interpretation process is very complex for "acceleration." For instance, very few students can use the specification of Example 4 to find the acceleration of a particle moving around an ellipse or spiral.

Concept-interpretation errors can arise either because the declarative specification itself is ambiguous or inconsistent, or because the interpretation process is defective. For example, students sometimes fail to invoke a concept specification which they know (relying instead on faulty intuitions or inappropriate bits of memorized information); or describe incorrectly a definition remembered by them; or often cannot devise a procedure to interpret a definition which they can correctly state.

### 2.2. Procedural Specification

This interpretation mode relies on knowledge defining a concept by an explicit procedure specifying how to identify or construct the concept. The stored knowledge about the concept is thus procedural rather than declarative. The corresponding interpretation process involves then (1) invoking the general procedural specification of the concept, (2) retrieving or elaborating its detailed description, and (3) implementing the procedure in the particular instance of interest.

The following examples restate in procedural form the declarative specifications of the concepts already discussed in Examples 2, 3, and 4.

**Example 5: Angle between two vectors.** The procedure specifying this concept consists of the following two major steps: (1) Draw the arrows, representing the two vectors, so that they start from the same point; and (2) Identify the angle (between 0° and 180°) enclosed by these arrows. This angle is called "the angle between the vectors."

**Example 6: Component vector.** The procedure specifying this concept consists of the following major steps illustrated in Figure 3c or 3d: (1) Consider the arrow representing the vector A of interest; (2) From the beginning of this arrow, draw a line parallel to the direction of interest; (3) From the end of this
arrow, draw a line perpendicular to \( i \); and (4) Draw the arrow from the beginning of \( A \) to the intersection point of these two lines. The vector represented by this arrow is called "the component vector of \( A \) along \( i \)."

**Example 7: Acceleration.** The procedure specifying this concept consists of the following major steps illustrated in Figure 4: (1) Identify the velocity \( v \) of the particle at the time \( t \) of interest; (2) Identify the velocity \( v' \) of the particle at a slightly later time \( t' \); (3) Find the small velocity change \( \Delta v = v' - v \) of the particle during the short time interval \( \Delta t = t' - t \). (Graphically, draw from the same point the arrows representing \( v \) and \( v' \); the arrow from the tip of \( v \) to that of \( v' \) represents then \( \Delta v \);) (4) Find the ratio \( \Delta v/\Delta t \), indicated by an arrow having the same direction as \( \Delta v \), but a different magnitude which (unlike \( \Delta v \)) is ordinarily not small; and (5) Repeat the preceding calculation, with the time \( t' \) chosen progressively closer to the original time \( t \), until the ratio \( \Delta v/\Delta t \) approaches a limiting value which no longer changes. This value is called the "acceleration \( a \) of the particle at the time \( t \) of interest." (Thus \( a = dv/dt \) if \( dv \) and \( dt \) denote the infinitesimally small differences \( \Delta v \) and \( \Delta t \).)

A procedural concept specification (or "operational definition") has the following major advantages: (1) It provides a more detailed and explicit specification of a concept than a declarative specification. Hence, it also helps to avoid imprecisions or inconsistencies which can easily creep into verbal definitional statements not subjected to tests of procedural implementation. (2) It provides an easier starting point for concept interpretation since one needs merely to retrieve an interpretation procedure which is already stored, rather than to generate one from a declarative concept specification.

The differences in accuracy and ease of concept interpretation are fairly small in a case, like that of "angle between two vectors," where the procedural specification of Example 5 is only a relatively minor elaboration of the declarative specification of Example 2. However, the differences become more pronounced in trying to find the "component vector" of \( A \) in Figure 3b, a task easier for students when starting from the procedural specification of Example 6 than from the declarative specification of Example 3. Finally, the procedural specification of "acceleration" in Example 7 makes much more apparent the substantial complexities hidden in the declarative

![Figure 4. Diagram illustrating the procedural definition of "acceleration."](image-url)
specification of Example 4. Indeed, even some individual steps of the procedural specification involve complex subprocesses (e.g., step 3 involves the vector subtraction of two velocities which may differ both in magnitude and direction, and step 5 involves a subtle limiting process). Thus, it is not surprising that most students cannot use the declarative specification of "acceleration" to determine the acceleration in particular situations (e.g., to determine that the acceleration of the pendulum bob in Figure 5a has the directions indicated in Figure 5b).

The disadvantages of a procedural concept specification are consequences of its very explicitness: (1) It is usually more lengthy than a declarative specification. (2) Its emphasis on interpretation details may obscure some important general features of a concept.

Although a procedural concept interpretation is easier to implement than a declarative one, it is not trivial. For example, errors may occur because one fails to invoke an available procedural specification, or implements incorrectly some of the steps specified by it.

2.3. Derived Specifications
A declarative or procedural specification, which provides the primary definition of a scientific concept and ultimate criterion for its proper identification, can be used to derive other specifications useful for interpreting the concept. Such a derived specification, even if not entirely equivalent to a primary one, may also be generally applicable. It can have the advantage of providing an easier starting point for interpreting a concept. But it may have the disadvantage of being further removed from elementary notions about the significance of the concept.

Example 8: Acceleration. By starting from the procedural specification of acceleration in Example 7, one can derive the following results describing the components of the acceleration of a particle moving along any path (com-
ponents due, respectively, to changes in the magnitude and direction of the velocity): The component along the velocity is $\frac{dv}{dt}$ (i.e., the rate of change with time of the magnitude $v$ of the velocity); the component perpendicular to the path, toward its concave side, is $v'/r$ (where $r$ is the radius of curvature of the path). This derived specification of the acceleration provides an easy starting point for identifying or finding the acceleration in many instances.

3. NONFORMAL CONCEPT INTERPRETATIONS

In science, as in everyday life, concepts can be interpreted in ways which are fast and effortless, although not designed to ensure maximal precision and generality. This section discusses some of these less formal case-based modes of concept interpretation.

3.1. Associated Knowledge Fragments

This mode of concept interpretation relies on the retrieval of stored fragmentary knowledge elements associated with a concept. Interpretation of the concept in a particular situation involves then usually little or no deliberate processing, nor any explicit invocation of a definition of the concept. Instead, more automatic recognition processes are often used to match some features of the given situation with features of available knowledge fragments.

*Example 9: Speed of a ball.* After being hit by a bat, a baseball travels along the trajectory illustrated in Figure 6. What is the speed of the ball at the highest point of its trajectory?

Some novice physics students and non-physics faculty members respond by saying that the speed is zero. When then asked "What is your definition of 'speed'?", some say, after some thought, that it describes how position changes with time. At that point, they spontaneously realize that their previous answer was incorrect and that the speed is not zero.

Note that, although the initial question is explicitly about "speed," it does not prompt any invocation of a definition of this concept—until the experimenter intervenes. Instead, the question is promptly answered by invoking some knowledge fragments associated with the notion of speed (perhaps some familiar knowledge about the speed of a vertically thrown ball at its highest point, knowledge that may also be cued by perceptual attention focused selectively on motion along the vertical direction).

![Figure 6. Trajectory of a baseball.](image-url)
Example 10. Acceleration. When asked about the acceleration of a vertically thrown ball at its highest point, or about that of a pendulum bob at the extreme end of its swing, many students quickly answer that the acceleration is zero. They do this by merely invoking an incorrect knowledge element alleging that the acceleration of a particle must be zero when its velocity is zero. Here again there is no attempt to retrieve or apply a definition of "acceleration," even when this definition can be stated. Correspondingly, there is also a failure to heed important discriminations implied by the definition, e.g., that the velocity of a moving particle can change even if its velocity happens to be zero.

Concept interpretation relying on associated knowledge fragments has the great advantage of being fast and effortless. The disadvantages are equally clear: (1) Frequently the knowledge fragments associated with a particular concept are ambiguous and vague, are not adequately discriminated from other similar knowledge elements, and are not accompanied by well-specified applicability conditions; (2) They are usually fairly incoherent, like much of everyday knowledge (diSessa, in press). Hence, they can easily be inconsistent and do not provide good bases for making general inferences; and (3) Because the interpretation process relies largely on recognition processes rather than more explicit procedures, it can be quite context-dependent, can be sensitive to salient surface features of a situation while ignoring others of greater significance, and can often fail to make important discriminations. These disadvantages lead frequently to concept interpretation errors, particularly when a scientific concept is applied in somewhat unfamiliar situations.

3.2. Standard Cases
This mode of concept interpretation uses knowledge stored in memory about specially singled out standard cases of the concept. The process, used to interpret the concept in a particular situation, then consists of (1) the retrieval of an appropriate standard case, and (2) a comparison used to match the given situation to this standard case. This comparison may involve either perceptual recognition processes or more explicit procedures. It may even be a complex multi-step problem-solving process whereby the comparison is made after transforming the given situation to one closer to a standard case.

When standard cases are not deliberately identified and the comparison is almost automatic, this mode of concept interpretation is similar to the way everyday entity concepts are commonly identified by comparisons with prototypes (Rosch, 1975, 1978). However, important standard cases, for entity as well as property concepts, are often explicitly selected and studied. For example, the useful role of standard examples in mathematics has been pointed out (Michener, 1978; Rissland, 1985).

Example 11: Resistors in parallel. When students are asked whether the resistors $R_1$ and $R_2$ are connected "in parallel" in the three situations illustrated in
Figure 1, all say properly that they are in parallel in Figure 3a. But some say incorrectly that they are not in parallel in Figure 3b, and many say incorrectly that they are in parallel in Figure 3c.

Students answer the questions, not by invoking a definition like that of Example 1, but by comparing all situations with Figure 3a which is the standard diagram used to illustrate resistors connected in parallel. It is then easy to understand the students' answers since Figure 3b looks perceptually different from Figure 3a, while Figure 3c looks quite similar.

**Example 12: Component vector.** What is the component vector of the vector A along the direction i? Most students answer this question easily when asked about the situation in Figure 3a; but they have more difficulties, or answer incorrectly, when asked about the situation in Figure 3b.

The reasons are quite clear. Students do not invoke a formal definition, like that of Example 6, but appeal instead to familiar standard cases where the direction of a component vector is either horizontal (as in Figure 3c) or vertical. The situation of Figure 3b is then troublesome because the direction i, along which the component vector is to be found, is at an odd angle. Indeed, it is fairly common to see students who actually rotate the piece of paper, with the diagram of Figure 3b, until i is horizontal. Thus, they use an auxiliary rotation to obtain a transformed situation which can then be directly compared with the familiar standard case of Figure 3c.

**Example 13: Acceleration.** The case of circular motion with constant speed (i.e., constant magnitude of velocity) is a classic standard case discussed in every physics course. There it is shown that the definition of acceleration (Examples 4 or 7) implies that the acceleration is directed toward the center and has a magnitude \( v^2/r \), where \( v \) is the speed and \( r \) is the radius. Expert physicists often resort to comparisons with this standard case to answer questions about the acceleration for ellipses or spirals (locally approximated by circles), and even about situations where the speed is a maximum or minimum (i.e., merely momentarily constant, rather than strictly constant).

Appeal to a standard case has the advantage of allowing quick and easy interpretation of a concept by comparison with a familiar case which is well specified and well understood. There are two main disadvantages: (1) Available standard cases may not be sufficient to cope with all the various instances where the concept needs to be interpreted; and (2) The process of comparison with the standard can easily be faulty if one relies on perceptual processes or analogical reasoning with ill-specified criteria of what constitutes an adequate match with the standard case. For example, important discriminations may not be made because significant features are not heeded (as illustrated by student responses in Figure 1c). Conversely, inappropriate discriminations may be made although they are irrelevant to the definition of a concept (as illustrated by student responses to Figure 1b). (3) Because of their saliency and familiarity, standard cases may often be retrieved indiscriminately when they are not applicable.
Example 14: Improper invocation of standard cases.

When asked about the acceleration of a particle moving around a circle, many students promptly say that the acceleration is directed toward the center. They do this without appealing to the definition of acceleration, but merely by invoking the familiar standard case of circular motion with constant speed—often applying this case inappropriately even if the speed is not constant.

When asked about the angle between the vectors A and B in Figure 2a, where the three vectors A, B, and C have the same magnitude, most students answer promptly (and incorrectly) that the angle is 60°. They do this without any attempt to invoke the definition of the concept “angle between vectors” about which the question was asked. Instead, they merely retrieve their standard geometrical knowledge about the angles in an equilateral triangle. (Many students correct their wrong answer after being asked to state a definition of “angle between vectors.”)

3.3. Classified Cases

This interpretation mode, which is a systematic extension of the preceding one, relies on stored knowledge comprising a classification of the domain of concept instances into a few distinct types of cases. This typology may even be summarized in the form of explicit rules. The process of interpreting the concept in a particular instance involves then (1) retrieving the classification scheme, and (2) matching the given instance to fit one of the existing types.

Example 15: Acceleration. The qualitative properties of the acceleration can be summarized by the following classification scheme, categorized according to different ways that the velocity of a particle can change for various kinds of motion: (The scheme can be inferred from the procedural specification of the acceleration in Example 7, or even more easily from the derived component specification in Example 8.) (1) If a particle moves along a straight path with constant, increasing, or decreasing speed, its acceleration is zero in the first case, non-zero and directed along the velocity in the second case, and non-zero and directed opposite to the velocity in the third case; and (2) If a particle moves along a curved path, its acceleration is non-zero and directed toward the concave side of the path. More specifically, if the particle’s speed is constant, increasing, or decreasing, the direction of its acceleration is perpendicular to its velocity in the first case; at an angle less than 90° in the second case; and at an angle greater than 90° in the third case.

A good classification into types of cases offers the following advantages: (1) It can be a very convenient starting point for quick and effortless concept interpretations; (2) It can make apparent important properties of a concept and thus reveal implications which are completely hidden by a formal definition. (For example, the properties of acceleration, listed in Example 15, are surprisingly rich implications of the innocent-looking definition of Example 4); and (3) If the classification is well-specified, the concept interpretation may attain a degree of unambiguity and precision approaching that
of more formal definitions. Indeed, classification can provide a powerful basis for problem solving (Clancey, 1985).

However, concept interpretation relying on classified cases has several disadvantages compared to more formal interpretations: (1) Unless the classification of cases is complete rather than partial, it does not allow reliable concept interpretation in all relevant instances; (For example, the classification scheme of Example 15 would be insufficient to determine quantitative values of the acceleration in various instances); and (2) A classification scheme does not provide the coherence of a single widely applicable concept specification which encompasses all possible cases in a unified way. Hence, it does not provide a good starting point for making general inferences and can thus lead to special kinds of errors.

Example 16: Incorrect inference about acceleration. "If the speed of a particle at some instant is zero, can its acceleration be zero?" In one of our protocols the subject answered this question by successively examining the acceleration for various possible cases of motion along straight lines and curves. He made correct statements about several difficult cases, but forgot to consider the simplest case where a stationary particle remains at rest. This failure to examine exhaustively all possible cases led him to the incorrect conclusion that the acceleration cannot be zero. (By contrast, appeal to a general definition of acceleration would have been a much easier starting point for making a general inference about this concept.)

More common errors arise because the classification scheme is incomplete or incorrect. For example, novice students' fragmentary knowledge does sometimes exhibit rudiments of a classification; but this classification is usually only partial and often includes incorrectly specified types of cases (e.g., the case that, if the velocity is zero, the acceleration must be zero).

4. "Ideal" and Actual Concept Interpretations

4.1. "Ideal" Concept Interpretation

The preceding discussion of various possible modes of concept interpretation provides the requisite background to address the following question: What interpretation mode or modes are optimally suited for interpreting scientific concepts? An answer to this question constitutes an "ideal model" of scientific concept interpretation. Such a model may be "prescriptive" rather than just descriptive (Heller & Reif, 1984) since it may be partly based on theoretical considerations, without necessarily mimicking what actual experts do or merely assuming that experts perform optimally. The formulation of such a model is of interest for several reasons: (1) It serves to identify what underlying knowledge and thought processes can lead to an effective and efficient interpretation of scientific concepts; (2) It can provide a useful basis of comparison for analyzing the observed concept interpretations of
expert scientists and novice students; and (3) It identifies essential knowledge and skills that need to be taught to improve science instruction.

To be effective in attaining the scientific goals of precise and widely applicable explanation and prediction, scientific concepts need to be adequately unambiguous, precise, consistent, and general. These criteria can be met by interpreting these concepts in the explicitly formal ways discussed in Section 2. In particular, procedural specifications provide the most explicit and reliable ways of interpreting scientific concepts. However, declarative specifications are useful because of their compactness, and derived specifications may facilitate ease of application.

Although formal methods are sufficient to interpret scientific concepts, their systematic implementation may be slow and cumbersome. Thus, one needs also be concerned with cognitive efficiency and limitations of human information-processing. In particular, the ability to deal with scientific problems requires that available mental capacity be predominantly devoted to more complex aspects of problem solving than the mere interpretation of relevant concepts (just as the ability to read complex prose requires predominant attention to comprehension tasks, rather than to the decoding of individual words and phrases). The nonformal concept-interpretation modes, discussed in Section 3, have the virtue of being cognitively efficient. In particular, concept interpretation can be quick and effortless if one has available a repertoire of compiled knowledge elements about various special cases and standard cases of a concept. Indeed, if knowledge in this repertoire is sufficiently familiar, it can be recognized and retrieved almost automatically without much conscious processing.

Such a repertoire of nonformal knowledge constitutes "intuitive scientific knowledge" which can be highly useful despite its intrinsic limitations of precision, accuracy, and coherent generality. However, it is essential that it have the following properties: (1) It must be consistent with formal scientific knowledge; (2) It must be testable against such formal knowledge when the need arises; and (3) It must incorporate discriminations needed to avoid confusions with nonscientific intuitive knowledge and with previously encountered scientific notions.

The preceding considerations suggest the following model for interpreting scientific concepts optimally: To achieve concept interpretation which is both reliably effective and cognitively efficient, the interpretation relies on both formal and nonformal interpretation modes used in complementary ways. If one encounters a familiar situation, it is most efficient to resort to one's repertoire of nonformal case-based knowledge. Doubtful conclusions can then be checked against more formal knowledge. On the other hand, if one encounters a situation which is unfamiliar, or runs into inconsistencies, or needs to make general inferences, it is best to resort to one's formal knowledge about the concept. Nonformal knowledge can then still be useful to provide checkpoints for more abstract arguments.
4.2. Observations of Actual Concept Interpretations

The preceding examination of possible and optimal ways of interpreting scientific concepts leads to the following question: What modes of concept interpretation are actually used by expert scientists and novice students? An extensive consideration of this question would be beyond the scope of this paper. The results of a detailed investigation, dealing with the concept "acceleration," will be published elsewhere; a brief account can also be found in Reif (in press). The following paragraphs merely summarize some salient results.

Most of these results are based on observations, made by me and my collaborators (Lisa Quinn and Peter Labudde), in which individual physics professors, or novice students enrolled in an introductory mechanics course, were asked to answer various questions about acceleration. To ensure conditions of minimal intrusiveness, each subject first answered all questions while being asked to talk out loud, but without further intervention by the experimenter. Afterwards, the subject was asked the same set of questions again, but with requests for fuller explanations of his or her answers. The transcript of each subject's tape-recorded utterances, together with his or her written work, constituted then a protocol which was subsequently analyzed in detail.

The following problem illustrates the types of questions asked about acceleration. (Similarly qualitative questions were asked about motions along ellipses, spirals, or straight lines, with constant, changing, or momentarily zero speed.)

Example 17: Pendulum problem. A pendulum bob, suspended by a string from the ceiling, swings back and forth as illustrated in Figure 5a. At the extreme point A of its swing, the speed of the bob is momentarily zero. As it descends with increasing speed along a circular arc, the bob passes the point B and attains its maximum speed when it is at the lowest point C where the string is vertical. Then the bob continues moving with decreasing speed, going through the point D and finally reaching the extreme point E where the speed of the bob is again momentarily zero.

At each of the points A, B, C, D, and E, is the acceleration of the bob zero or not? If not, draw an arrow indicating its direction.

4.3. Experts' Concept Interpretations

The concept interpretation of "good experts" (e.g., of physics professors who performed well in answering questions about acceleration) resembles that of the ideal model discussed in Section 4.1.

In particular, such experts commonly interpret concepts by using both formal definitional knowledge and less formal compiled knowledge about special cases. For example, when asked about the acceleration of the pendulum bob at the points A, B, C, D, and E in Figure 5a, a typical expert answered first the question about the point C because he could then imme-
diately apply his special knowledge about acceleration for the standard case of circular motion with constant speed. When turning his attention to point A, the expert had no such special knowledge available; hence, he reverted to the formal definitional procedure of comparing explicitly the velocity at this point and at a neighboring point. Finally, he found the accelerations at the remaining points by symmetry or interpolation arguments, and thus obtained the correct answers illustrated in Figure 5b.

Experts' compiled knowledge and intuitions, acquired by dint of years of experience, may be insufficient to deal even with very elementary concepts and familiar situations. (The reason is that such compiled knowledge is quite situation-specific.) For example, acceleration is a very elementary concept, learned at the beginning of any introductory physics course; a pendulum is also a very familiar object repeatedly discussed throughout physics. Nevertheless, we have never yet encountered an expert physicist who did not have to use some explicit reasoning to answer the qualitative questions about the pendulum problem of Example 17.

Experts often appeal explicitly to general knowledge and then elaborate it for application to particular situations. For example, to solve the pendulum problem of Figure 5a, some expert physicists apply the procedural definition of acceleration at points A and B, and occasionally even at C; and quite a few apply systematically at all points the derived formal knowledge (Example 8) about the components of acceleration.

Finally, it is worth noting that not all experts are "good experts." Indeed, if one selects experts by using merely nominal criteria (such as titles, degrees, positions, or amount of experience), rather than actual performance criteria, one can easily find "nominal experts" whose performance is reminiscent of that of novices. Thus, we have encountered physics faculty members, employed at good universities and recently engaged in teaching introductory physics courses, who performed fairly poorly in interpreting the acceleration concept which they had recently taught.

The preceding observations suggest a warning: Cognitive studies, where experts are merely selected on the basis of nominal criteria, may not provide trustworthy information about thought processes leading to good intellectual performance. However, it may be illuminating to study how performance deficiencies of experienced people can be traced to particular deficiencies of their underlying knowledge and thought processes.

4.4. Novices' Concept Interpretations

Novice students are often quite poor at interpreting scientific concepts. For example, the students observed by us had actively used the concept "acceleration" for a couple of months in an introductory college-level mechanics course; nevertheless, they answered incorrectly more than 50% of our qualitative questions about acceleration. [Such results are consistent with other data about students' interpretation of acceleration (Trowbridge & McDer-
Students' modes of concept interpretation also differ markedly from ideal ones (Section 4.1) or from those of good experts.

The main characteristics of students' concept interpretation can be summarized as follows: (1) Students' conceptual knowledge is highly incoherent, consisting largely of various knowledge fragments. Furthermore, many of these are incorrect because of inadequate discriminations, or because of confusions with preexisting knowledge acquired in daily life or prior science instruction; (2) Such knowledge fragments are often retrieved fairly quickly and relatively little processing is done afterwards. (3) Relevant applicability conditions are often ignored; (4) General definitions are rarely invoked, and, if invoked, can usually not be properly interpreted; (5) The knowledge organization is flat and undifferentiated, i.e., no particular knowledge elements are signaled out as more fundamental or important than any others; (6) Inconsistencies are often encountered, but perceived paradoxes cannot be satisfactorily resolved because the underlying knowledge is too fragmented; and (7) Students themselves often realize that they are confused and that their conceptual knowledge does not "hang together."

The following are some examples of typical knowledge fragments invoked by students: “If the speed changes, the acceleration can't be zero” (correct). “If a particle travels with constant speed, its acceleration must be zero” (incorrect in general, but consistent with everyday notions of acceleration). “If the velocity of a particle is zero, its acceleration must be zero” (this ignores the discrimination between velocity and change of velocity, and is incorrect unless the velocity is constant). “If a particle travels around a circle, its acceleration is directed toward the center” (this result is merely remembered from science instruction, can rarely be justified by the student, and is often invoked incorrectly by ignoring that it is only applicable if the speed is constant).

Appeal to such fragmented knowledge results in wrong answers to most of the questions about the pendulum in Figure 5a. Although many students can correctly state the definition that “acceleration is the rate of change of velocity,” or write down the equivalent formula \( a = \frac{dv}{dt} \), most never apply it. If advised to apply this definition in the pendulum problem of Figure 5a, they cannot interpret the definition and thus obtain no help from it. Thus, some students claim incorrectly that the acceleration of the pendulum bob at point A is zero (since the velocity there is zero), while stating a definition which clearly implies otherwise.

5. DISCUSSION AND IMPLICATIONS

5.1. Learning Difficulties
There are well-recognized reasons, apparent from past investigations like those cited in Section 1, why the learning of scientific concepts is difficult: Novel scientific concepts and conceptual frameworks need to be learned;
they need to be applied instead of long-familiar everyday concepts and frameworks; and they need to be carefully discriminated from such preexisting concepts, despite surface similarities and occasionally even identical names. These difficulties can easily result in scientific misconceptions, to confusions with preexisting concepts, and to inappropriate reversions to the use of traditional everyday notions.

But these difficulties, although substantial, are transcended by more pervasive and subtle difficulties apparent from the discussion of the preceding pages. It is not just that various preexisting concepts need to be replaced by new ones, without attendant confusions. It is that the form, rather than just the content, of the new knowledge is significantly different.

Although science is, in the words of Einstein, "nothing more than a refinement of everyday thinking," this refinement can be considerable and gives rise to some major differences between everyday concepts and scientific ones. Thus, everyday concepts are used with the implicit goal of ensuring satisfactory human functioning in daily life. This goal can be attained with concepts which may be somewhat ambiguous and vague, occasionally inconsistent, and limited in scope. Hence, it is usually sufficient to specify the meanings of everyday concepts implicitly by the contexts in which they are used, or to identify concepts by implicit comparisons with prototypes (Rosch, 1975, 1978).

By contrast, scientific or mathematical concepts are used to pursue the explicit scientific goal of achieving optimal predictive and explanatory power. To achieve this goal, one must ensure that concepts can be specified with minimum ambiguity, maximum precision, maximum consistency, and highest generality. Correspondingly, one needs also to resort to thought processes, like those discussed in Section 3, which are more explicitly formal and complex than those ordinarily used in daily life.

These differences give rise to the following learning difficulties, particularly for students previously unfamiliar with scientific ways of thinking: (1) The need for unambiguity and precision requires fine discriminations; (2) Careful use of language and other symbol systems is required to ensure that all symbols are unambiguously related to their referents and to each other; (3) Concepts must be specified abstractly to achieve generality, but with procedural knowledge ensuring their unambiguous interpretation in any specific instance; (4) The entire knowledge must be coherent and consistent; and (5) It is important to develop intuitive scientific knowledge which can be used quickly and effortlessly. But this knowledge must be consistent with formal scientific knowledge, must be testable against it, and must be automatically discriminated from everyday intuitions.

In addition to these cognitive difficulties, students face metacognitive difficulties. Thus, they approach the learning of science not only with naive mental models about the physical world, but also with naive conceptions about science and human thought processes—and such misleading concep-
tions can crucially determine how students direct their attention and what they learn (Schoenfeld, 1983). For example, students find it difficult to believe that familiar concepts used in everyday life can be scientifically meaningless, or that new scientific concepts may be freely invented—provided that this is useful and can be done consistently. Students often think of science as a collection of facts and formulas; correspondingly, they try to learn new scientific concepts by merely memorizing associated facts and verbal definitions. Being unaware of cognitive issues like those discussed in the preceding pages, they don't realize that one must be able to interpret a definition, must develop intuitive scientific knowledge, and must ensure that all this knowledge is coherent and flexibly usable. Correspondingly, students often have little inkling of the kinds of active learning efforts required to acquire new scientific concepts as effective thinking tools.

5.2. Instructional Implications
Section 3, which discussed optimal concept interpretation and students' observed deficiencies, suggests the following guidelines for teaching scientific concepts more effectively: (1) Teach students explicit concept specifications, especially concept-interpretation procedures; (2) Let students actively implement such interpretation procedures in various typical and error-prone cases, thereby helping them compile a repertoire of knowledge about special cases and common errors; (3) Encourage intuitive use of such acquired knowledge in commonly occurring situations; and (4) Make sure that students' entire knowledge about a concept is coherent and consistent, and that they can check whether their concept interpretations are correct.

Common teaching practices deviate appreciably from such guidelines. For example, new concepts are often introduced by descriptive statements or vague analogies, but explicit interpretation procedures are rarely taught. Since knowledge about important special cases is often merely presented, rather than generated by the students themselves, it remains fragmented. Furthermore, little effort is made to teach students how to detect and diagnose their misconceptions and concept-interpretation errors.

A subset of these instructional guidelines was recently explored by us in an instructional experiment (Labudde, Reif, & Quinn, in press) which, during a limited intervention lasting about half an hour, did merely the following: (1) It taught students an explicit interpretation procedure for the concept "acceleration" (a procedure like that of Example 7); and (2) It provided students with explicit practice in detecting, diagnosing, and correcting their own errors and those allegedly committed by someone else.

The following were the main outcomes of this experiment: (1) Students' ability to interpret the concept "acceleration" improved markedly (from 40% of questions answered correctly on a pretest, to 95% of questions on an equivalent posttest); (2) Students invoked many incorrect knowledge elements during the pretest, but almost none on the posttest; (3) After the in-
struction, students invoked correct knowledge elements to answer simple questions, but used the procedural definition of acceleration to answer more complex questions or to check answers about which they felt uncertain; and (4) Although the instruction did not provide explicit guidance about how to diagnose errors, it was sufficient to enable students to detect concept-interpretation errors and to identify appropriate reasons why they had been committed.

Although this experiment was quite limited in scope, it does provide evidence for the efficacy of teaching methods designed to ensure that students acquire conceptual knowledge which is explicit and coherent, which can be procedurally interpreted, and which can be monitored for correctness.

5.3. Limitations and Extensions
The discussion in this paper has dealt only with basic interpretation tasks needed to identify or construct specific instances of scientific concepts. As the preceding pages have indicated, such basic tasks are far less simple than one might naively suspect and cause many difficulties for students. They are also an essential prerequisite for any scientific work.

By addressing first such relatively simple aspects of scientific concept interpretation, we have tried to identify a few central cognitive issues and their implications. The simple instructional experiment described in Section 5.2 (Labudde, Reif, & Quinn, in press) was carried out in this spirit and showed that mere attention to a few such issues can produce some marked effects.

The analysis of this paper paves the way for studying some more complex aspects of concept interpretation, such as the important role of symbolic representations, the application of concepts to inferencing tasks beyond mere concept instantiations, and the interactions between concepts in a broader conceptual framework.

This analysis can also serve as the basis for formulating instructional models for teaching scientific concepts effectively. We are currently trying to implement such models in computational environments where learning conditions can be well controlled and significant parameters can be deliberately varied. We hope thereby to provide a testbed for studying experimentally various cognitive questions concerning the interpretation and learning of scientific concepts. Furthermore, such work has obvious practical applications for devising more principled and effective methods for teaching scientific concepts.

REFERENCES


