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# Physically Distributed Learning: Adapting and Reinterpreting Physical Environments in the Development of Fraction Concepts

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**Abstract**

Five studies examined how interacting with the physical environment can support the development of fraction concepts. Nine- and 10-year-old children worked on fraction problems they could not complete mentally. Experiments 1 and 2 showed that manipulating physical pieces facilitated children's ability to develop an interpretation of fractions. Experiment 3 demonstrated that when children understood a content area well, they used their interpretations to repurpose many environments to support problem solving, whereas when they needed to learn, they were prone to the structure of the environment. Experiments 4 and 5 examined transfer after children had learned by manipulating physical pieces. Children who learned by adapting relatively unstructured environments transferred to new materials better than children who learned with "well-structured" environments that did not require equivalent adaptation. Together, the findings reveal that during physically distributed learning, the opportunity to adapt an environment permits the development of new interpretations that can advance learning.

*Keywords:* Action; Mathematics learning; Fractions; Manipulatives

## 1. Introduction

How do physical actions contribute to abstract learning? Folk wisdom has it that there is something nearly magical about "the concrete"—if people see and touch, they will learn. The common shift from concrete to abstract reasoning nourishes this belief. Children can often solve problems in physical situations before they can succeed with symbolic representations (Bruner, Olver, & Greenfield, 1966; Piaget, 1953). The existence of this shift, however, does not entail that concrete behavior is responsible for the change. One might imagine that

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the acquisition of verbal ideas, rather than the performance of concrete actions, causes the shift.

There is surprisingly mixed evidence that concrete manipulation improves learning in educational settings (Chao, Stigler, & Woodward, 2000), and under some conditions, it can even interfere (Uttal, Scudder, & DeLoache, 1997). Random hands-on activities are no panacea for educational woes. At the same time, specific forms of physical action can support symbolic learning (Sowell, 1989). For example, touching objects helps children count accurately (Alibali & DiRusso, 1999; Case & Okamoto, 1996), and manipulating computer graphs can help children develop concepts of two-dimensional space (Sarama, Clements, Swaminathan, McMillen, & Gonzalez Gomez, 2003). This article examines whether actions can support abstract learning when they provide a way for children to adapt and then reinterpret their environment. We call this process *physically distributed learning* (PDL).

## 2. Research on learning through physical action

Physical action may support learning in a number of ways. The literature on embodied cognition (Barsalou, 2003; Gibbs & Berg, 2002; Glenberg & Robertson, 2000; Johnson, 1987), for example, argues that the semantics of symbolic thought grow from the possibility of physical action. For example, Lakoff and Núñez (2000) proposed that children's mathematical concepts (e.g., sets) develop through metaphorical extension of perceptual–motor relations (e.g., containership). In this work, our focus is more restricted. We want to know how taking physical action impacts thinking and learning. Fig. 1 offers four ways physical actions can support thinking and learning. We differentiate these situations by the degree of stability in ideas and environments.

In *induction* (Quadrant 1), people do not have stable, mature ideas, but they are operating in a well-structured and stable environment. Stable environments offer clear feedback and strong

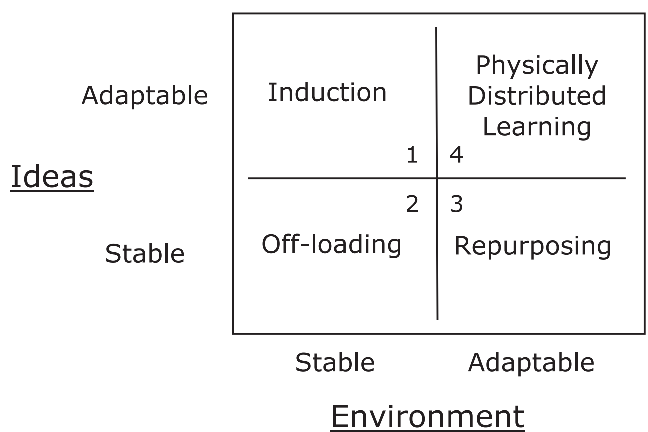


Fig. 1. Physical actions and learning. We distinguished four ways actions could support learning. Ideas and environments can be stable or adaptable. We presume these dimensions are continuous, but represent them in discrete quadrants for clarity in discussion.

constraints on what counts as a correct interpretation. These consistencies help people uncover the structural regularities in these environments through physical activity (Greeno, 1988). For example, pouring water back and forth between wide and thin glasses leads children to eventually discover that the quantity of liquid is invariant under transformation (Piaget, 1966). There are many forms of induction. From an ecological perspective, perceptual–motor activity helps people notice useful structures that remain stable as they move objects (Gibson, 1986). For example, handling a sheet of paper helps people notice that it maintains a rectangular shape despite changes in perspective. From a hypothesis testing perspective, physical actions enable people to query the environment to test their ideas (Klahr & Dunbar, 1988; Kuhn et al., 1988). Inductive learning is improvement in the individual's knowledge or ability to perceive (Gibson & Gibson, 1955).

In *off-loading* (Quadrant 2), people operate in a stable and often specialized environment, and they have stable ideas. In these situations, people rely on the environment to reduce the cognitive burden of a task. For example, highly trained pilots and the well-designed environment of a cockpit form a system that achieves complicated feats such as remembering airspeeds (Hutchins, 1995b). Activity in this quadrant is often referred to as *distributed cognition* (Hutchins, 1995a; Norman, 1988; Zhang & Norman, 1994). In these situations, people have stable interpretations of a felicitous environment and the desired end state. Learning in Quadrant 2 is an increase in the individual's efficiency at off-loading cognitive tasks to the environment.

In *repurposing* (Quadrant 3), people have stable ideas, but the environment does not have an ideal form. If people's ideas are mature enough, they can change the environment to achieve their goals (Kirsh, 1996). For example, in the computer game Tetris, players use keystrokes to translate and rotate falling pieces so they fit neatly into rows. As players improve, they learn to repurpose the keystrokes so that the movements of the pieces also yield information about where they will fit (Kirsh & Maglio, 1994). We assume there is a continuum of adaptable situations. For example, Tetris permits a relatively small adaptation, at least compared to people repurposing a knife to turn a screw, or inventing a screwdriver in the first place. Learning in Quadrant 3 is changes to the environment that permit people to implement their ideas more effectively.

Finally, in PDL (Quadrant 4), the environment and people's ideas are both adaptable. Although people do not change their ideas and environments constantly, adaptation can often be helpful. We propose that the emergence of new interpretations through physical adaptations of the environment can be an important benefit of physical action for learning abstract ideas.

For example, imagine a young child who is asked to create a one-fourth share of eight candies. Even though the child may not know anything about fractions, the child will have some interpretation of the problem. The child might view the situation as involving whole numbers and think that getting a *one*-fourth share of eight candies means getting *one* candy. To solve the problem correctly, the child will have to reinterpret what is already "known" (Karmiloff-Smith, 1992). The child will have to relax the whole quantity interpretation, which is highly practiced, and develop a new fraction interpretation.

Reinterpretation is difficult (Clement, 1993; Luchins, 1942; Vosniadou & Brewer, 1992). It is particularly difficult to achieve through thought alone. For example, after people developed an interpretation of an ambiguous figure (it could be interpreted as a duck or a rabbit), they could not produce a different interpretation when asked to do so with their eyes closed (Chambers & Reisberg, 1985). Their original interpretations shaped their memory for the picture and interfered with their ability to find a new one. Physical activity may help people adapt the environment and thereby facilitate reinterpretation. People can interact with their environments without exactly knowing the appropriate steps or envisioning the final state (O'Hara & Payne, 1998; Shirouzu, Miyake, & Masukawa, 2002). For example, in the previous fraction problem, the child might haphazardly push two pieces together. This new arrangement may help the child reinterpret the two pieces as "1 group" of pieces. As we describe in the following section, this reinterpretation of what is countable (from pieces to groups) can put the child on a trajectory to learn that one-fourth of eight means one of four groups of two. We label this process PDL to indicate that the learning is distributed between functional adaptations to the environment and the individual, and that early learning is not located in either alone.

## 3. PDL for fraction concepts

By late elementary school, 9- and 10 year-old children have developed strong ideas about whole numbers—they can fluently count and add. Fractions, however, require new interpretations. Researchers have identified several interpretations of rational numbers, including part–whole, ratio, quotient, and operator (e.g., Carpenter, Fennema, & Romberg, 1993; Confrey & Smith, 1995; Kieren, 1995; Moss & Case, 1999; Streefland, 1993; Thompson & Saldanha, 2003). Our purpose here is not to focus on one of these schemas, but to examine whether physical activity can contribute to the development of a viable fraction interpretation.

Two physical actions that children take with manipulatives are *unitizing* and *partitioning.* Unitizing involves treating objects or collections of objects one-by-one, for example, by pointing to each object or collection separately (Behr, Harel, Post, & Lesh, 1993; Lamon, 2002). Partitioning involves equally subdividing a collection of pieces (Behr et al., 1993). Partitioning and unitizing are simple physical actions. They are not explicit interpretations of quantity. Children often engage in these actions casually without much of a plan. Nevertheless, these actions may help children develop interpretations critical to the concept of fractions. In particular, they may help children reinterpret partitions as groups that can be unitized in their own right.

To see how this might work, Fig. 2 offers some typical adaptations and interpretations children develop for the problem "make ¼ of 8." In Fig. 2a children created two partitions from the total collection of pieces, and they unitized the pieces within each partition. They interpreted the 1 and 4 in ¼ as whole numbers with no particular relation. In Fig. 2b, children partitioned the collection into two equal piles of four pieces. They interpreted the circled pile as the 1 in ¼ and the four pieces as the 4. They were close to an interpretation of groups, because they counted the partition as a single unit. However, they did not relate this pile of four to the other pile, so they had not quite made a full interpretation of groups as subsets. In Fig. 2c, the children successfully partitioned the pieces into four equal groups and interpreted the answer to ¼ of 8 as the two pieces that make up the group. Our hypothesis is that physical manipulation supports this reinterpretation into groups.

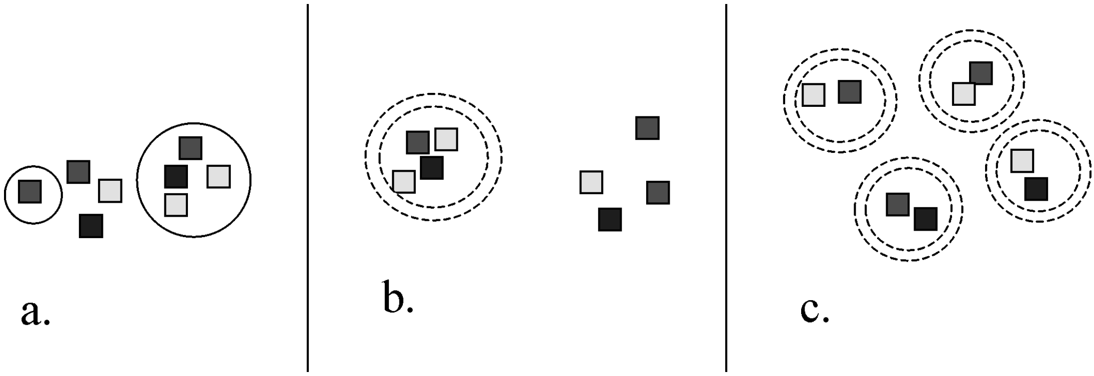


Fig. 2. Typical configurations and interpretations for the problem of "Make ¼ of 8." Panel 2a shows two collections, one of one piece and one of four pieces, that are interpreted as two separate whole numbers. 2b shows a partition into two groups of four. The interpretation is that one group of four pieces is one-fourth of the total. 2c shows partitioning into four equal groups. In 2c, each partition is correctly interpreted as one group of two pieces.

## 4. Experimental Overview

Five studies examined children's physical actions with *manipulatives* and the payoff those actions have for their learning about fractions. Manipulatives are small objects, such as pie wedges, frequently used in early mathematics instruction. The top-level goal of the research was to demonstrate the reality of PDL where learning involves both physical adaptation and reinterpretation (Quadrant 4). We show that children can solve fraction problems by adapting their environment and their own interpretations when they do not have an internal or external algorithm for the problems.

The first study shows children can solve problems when they move pieces, but not when they can only look at the pieces. This supports the importance of physical action. The second study shows that physical action supports reinterpretation. Children receive problems with the pieces prearranged into the end state (i.e., grouped). Children still perform better and develop the correct interpretation when they move the pieces, whether or not they are prearranged. This implies that a preexisting fraction interpretation does not drive their physical adaptations; otherwise, they should have recognized the end state. Instead, the reinterpretation emerged via the physical interaction. We offer one possible account of the process that leads to the reinterpretation through physical adaptation for these specific problems.

The third study shows the difference between distributed problem solving and distributed learning. Children receive problems in Quadrant 3 where they have a strong schema but an imperfect environment. In this case, they can impose their interpretations and repurpose the environment to solve the problems even though they cannot solve the problems in their head. In contrast, on poorly understood problems, the structure of the environment drives any useful actions.

Finally, the last two studies tested an implication for transfer. This implication is that PDL learning yields the flexibility necessary to leverage new environments, whereas inductive learning does not. Many educational tasks provide materials designed to help students solve particular problems leading to the inductive learning of Quadrant 1. This learning creates a trajectory toward Quadrant 2 where people can efficiently off-load cognition to stable environments. However, inductive learning is often dependent on the specific structures of the learning environment. Therefore, if the structures are unavailable, children will be unable to use what they have learned. In other words, too much reliance on well-structured environments can support performance at the expense of the insight that helps people develop understanding in new situations (Gilmore, 1996). PDL, in contrast, requires children to adapt the environment and their interpretations. Assuming that learners can make any headway at all, this process helps them generate critical structural features rather than becoming dependent on the environment to provide them. The interpretations that arise in PDL can put children on a trajectory to Quadrant 3 where they can impose stable ideas on new environments to repurpose them in useful ways.

## 5. Experiment 1: Effects of action

Experiment 1 tested whether manipulating physical materials affected children's development of viable interpretations of fractions in operator problems. Operator problems are like "make ¼ of 8" (Behr et al., 1993). For this problem, a correct interpretation of the fraction ¼ is "2." In contrast, an incorrect whole-number interpretation of ¼ might be "1" or "4." In the physical condition, children moved tile and pie pieces with their hands to help solve the problems (see Fig. 3). In the pictorial condition, the same children used a pencil to draw on pictures of pieces. We predicted that the process of physically moving the pieces would help children create physical partitions and interpret them as groups, which is a key step to finding the fraction interpretation. To test this prediction, we examined the children's interpretations and their physical organization of the pieces.

### *5.1. Methods*

#### *5.1.1. Participants*

Thirty-two 9- and 10-year-old children were randomly selected from two fourth-grade classes at the end of the school year. They had received prior instruction with manipulatives for how to add simple fractions (e.g., ¼ plus ¼). (See Experiment 5 for more details on the prior instruction.) They had not done operator problems.

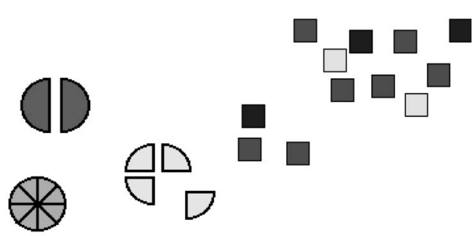

Fig. 3. Pie and tile manipulatives. Materials used in Experiments 1 to 5.

#### 5.1.2. Materials

Children used pie wedges and tiles to solve the problems (Fig. 3). Tiles are randomly colored squares. Pies wedges are circles divided into halves, 3rds, 4ths, 5ths, 6ths, 8ths, 10ths, and 12ths. Each division is a different color (e.g., halves are pink, thirds are orange). The pictorial materials were line drawings of pie or tile pieces plus a pencil. For each problem, the pieces were spread haphazardly across the table or picture.

#### 5.1.3. Design

The pictorial and physical materials made up the materials factor in a within-subjects design. For each material, children completed a problem with pies and a problem with tiles. Order of problem presentation was counterbalanced. The operator problems were interspersed among other questions. The quantities for each problem differed (e.g., $\frac{1}{3}$ of 9, $\frac{1}{4}$ of 8). The primary dependent measures were children's verbal accuracy (interpretations) and their arrangement of the pieces (adaptations) plus the number of moves they took and the number of times they spontaneously restarted from scratch on a problem.

### 5.2. Procedure

Children were videotaped as they worked individually with an interviewer. For the physical materials, the interviewer provided the correct number of pieces and asked the children to show a fraction of those pieces. For example, the interviewer might give a child 12 pieces and ask her to show ¼ of them. For the pictorial materials, the interviewer would give the child a picture of 12 pieces and a pencil. The interviewer gave no feedback and no time limit. Children stopped working on each problem when they decided they had finished and gave their final answer.

#### 5.2.1. Coding

In this and the following experiments, each solution received an *interpretation score* that reflected the child's verbal answer and an *adaptation score* that reflected the child's physical arrangement of the pieces. Adaptations and interpretations are not the same thing. For example, a child could arrange the pieces well, but give the wrong verbal answer. In Fig. 2c, one child correctly partitioned the pieces, but gave the verbal answer "1" to the question of $\frac{1}{4}$ of 8 pieces. Conversely, a child could give the correct answer without using the pieces.

We counted an interpretation as correct if the child stated the correct numerical answer. We coded an adaptation as partitioned if the child divided the entire set of pieces into equal groups. Equal sharing activities can help children develop fraction interpretations (Empson, 1999; Streefland, 1993), and the activity of creating equal partitions could support this development. The coding scheme was developed to address that possibility. The left side of Fig. 4 shows partitioned adaptations for the physical and the pictorial materials. Any other configuration that a child created was a nonpartitioned adaptation. The right side of Fig. 4 shows some possibilities for how children created adaptations that were not partitioned.

A primary and a secondary coder checked reliability on the adaptation codes using a subset (10%) of the students' responses drawn randomly from the interviews. Interrater agreement for the adaptation coding was 100% for this experiment and no less than 92% for any of the subsequent experiments. The primary coder subsequently scored all of the students' adaptations.

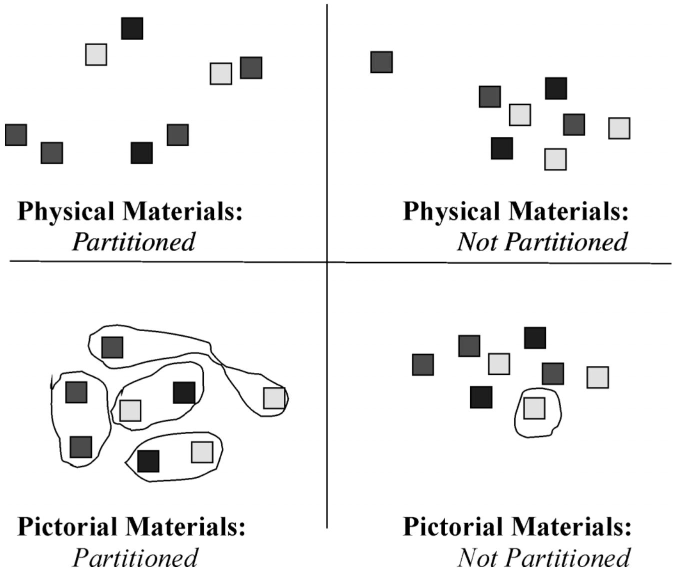


Fig. 4. Examples of partitioned and nonpartitioned adaptations of ¼ of 8. The left panels show partitioned adaptations created with physical materials and with pictures. In both cases, the set of eight pieces is divided into four equal groups of two. The right panels give examples of nonpartitioned adaptations. In both cases, the set is not partitioned equally, and instead one piece is isolated.

We also gathered two process measures. One measure indicated the children's overall physical activity by tallying the number of movements they made in reference to the materials. These movements included touching, pointing, drawing, shifting, covering, combining, and separating physical pieces. For the pictorial and physical pieces, children could make many of the same moves (e.g., touching, pointing), but moves such as shifting, combining, and separating worked differently. For example, a child could draw a circle around a group of pieces to indicate combining them. A movement refers to an event; touching and moving a piece counted as a single movement rather than two discrete movements. For example, if a child counted out four tiles by moving them one at a time, this was four moves; however, if a child moved the tiles as a group, this was one move. The second measure indicated when children restarted a problem by "erasing" the workspace and trying again. For example, a child could cross out or erase all the drawn lines on the picture or, in the physical condition, collect all the pieces to start again. These measures are only a rough index of level of activity. Undoubtedly, children are engaging in more cognitive activity in both the physical and pictorial conditions than these behavioral measures capture. This may be particularly true for children in the pictorial condition, as pencil and paper may be more difficult to manipulate than pieces.

### *5.3. Results*

For the same children in the same session on the same class of problems, physically manipulating the pieces had strong benefits compared to drawing on pictures of the pieces. Table 1 shows the percentages of children who gave accurate interpretations and created partitioned

Table 1
Percent of good interpretations and adaptations by materials

| Materials | Correct Interpretations | Partitioned Adaptations |
|---|---|---|
| Physical | | |
| Pies (%) | 59 | 64 |
| Tiles (%) | 81 | 68 |
| Mean (%) | 70 | 66 |
| Pictorial | | |
| Pies (%) | 19 | 4 |
| Tiles (%) | 13 | 0 |
| Mean (%) | 16 | 2 |

adaptations in each condition. We first describe the measures separately and then consider their relation to one another.

### 5.3.1. Measures taken separately

We found no significant effects involving tiles versus pies, so we combined the two for the statistical analyses. For both the pictorial and physical conditions, a maximum interpretation score was 2 points (correct for both the tile and pie problem), and a maximum adaptation score was 2 points (partitioned for both the tile and pie problem). Four students' videotapes were unusable for both the physical and pictorial problems. One additional student's videotape was missing for the pictorial problems. Consequently, 27 students were included in the following analyses. In a fully within-subjects analysis, children's interpretations were more accurate with the physical materials than with the pictures, $M = 1.4$, standard error [$SE$] = 0.14 and $M = .33$, $SE = .13$, respectively; $F(1, 26) = 37.1$, $MSE = 0.42$, $p < .001$. Similarly, children created partitioned adaptations more frequently with the physical materials than with the pictorial materials, $M = 1.3$, $SE = 0.16$ and $M = 0.04$, $SE = 0.04$, respectively; $F(1, 26) = 64.8$, $MSE = 0.3$, $p < .001$.

The children were also more active with the physical materials. The average number of moves per physical problem was 22.2 ($SE = 2.1$) compared to 9.1 ($SE = 1.6$) per picture problem, $F(1, 26) = 24.6$, $MSE = 89.3$, $p < .001$. Notably, the number of moves for the physical condition was well above the minimum number of moves needed to solve a problem. For example, a partitioned adaptation for ¼ of 8 requires four moves with the physical or pictorial materials (i.e., move or circle two pieces at a time into four separate groups). Thus, children were not executing an efficient plan to solve the problems in the physical condition. Similarly, children in the physical condition frequently started over by "erasing" the space. Children restarted more times in the physical condition than in the pictorial condition, $M = 1.7$, $SE = 0.13$ and $M = 1.1$, $SE = 0.04$, respectively; $F(1, 26) = 20.5$, $MSE = .3$, $p < .001$.

### 5.3.2. Relations between measures

In the physical condition, children gave more correct interpretations, made more partitioned adaptations, moved the pieces more, and tried more strategies. At the same time, within a condition, children's interpretation scores did not correlate significantly with their adaptation scores (physical, $r = .03$, $p = .88$; pictorial, $r = .2$, $p = .33$), their number of moves (physical, $r = -.13$, $p = .51$; pictorial, $r = -.00$, $p = .99$), or their number of restarts (physical, $r = .12$, $p = .53$; pictorial, $r = -.17$, $p = .41$). Thus, being able to move the pieces in the physical condition helped children but did not guarantee success. The lack of a strong correlation between adaptations and interpretations within a condition implies that the ability to adapt the environment and the ability to interpret the environment do not stem from a single underlying schema for these problems.

To further explore the relation between interpretation and adaptation, we aggregated the data a second way. For each problem, children sometimes restarted. By ignoring problem boundaries, we considered both aborted and final attempts as separate "tries" to get a more refined account of change over time. Most children completed three total tries with the physical materials and two with the pictures. Table 2 shows the data for the first, second, and third attempts with the physical problems and the first and second attempts with the pictorial problems. The table shows the percentage of tries that ended in partitioned and nonpartitioned adaptations and the percentage of correct interpretations for each. (Recall that children did not receive feedback, so they could make a correct interpretation but still decide to restart on the problem). Table 2 also shows the number of moves children made when they created partitioned and nonpartitioned adaptations.

Creating a partitioned structure did not guarantee interpretive success. When children made partitioned adaptations with the physical materials, they still gave incorrect interpretations approximately 40% to 70% of the time. Conditional probabilities help clarify the implications of this finding. When we collapsed the physical and pictorial conditions, a partitioned structure yielded a .44 probability of making a correct verbal answer, whereas a nonpartitioned structure yielded a .22 probability of making a correct verbal answer. The chance of making a partitioned structure was .56 in the physical condition and .02 in the pictorial condition. Thus, the

Table 2
Interpretations and moves broken out by adaptation over problem-solving tries

| | | | Moves (*SE*) | |
|---|---|---|---|---|
| Try | Adaptations | % Correct[a] | Correct | Incorrect |
| Physical materials | | | | |
| Try 1 ($n = 28$) | partitioned 64% | 33% | 18.5 (4.4) | 13.0 (2.1) |
| | ~ partitioned 36% | 20% | 9.0 (1.7) | 8.8 (1.4) |
| Try 2 ($n = 28$) | partitioned 46% | 60% | 17.7 (4.0) | 16.3 (3.5) |
| | ~ partitioned 52% | 46% | 7.8 (2.0) | 9.0 (2.1) |
| Try 3 ($n = 21$) | partitioned 52% | 36% | 12.3 (1.7) | 17.4 (2.4) |
| | ~ partitioned 48% | 30% | 13.7 (3.8) | 15.4 (3.1) |
| Pictorial materials | | | | |
| Try 1 ($n = 27$) | partitioned 0% | — | — | |
| | ~partitioned 100% | 11% | 10.0 (1.4) | 9.7 (1.5) |
| Try 2 ($n = 27$) | partitioned 4% | 100% | 7.0 (n/a) | — |
| | ~partitioned 96% | 19% | 5.4 (2.1) | 8.3 (3.3) |

[a]Percentage of correct interpretations within partitioned/nonpartitioned adaptations.

physical materials helped the students make more partitioned structures, which presumably improved interpretation. At the same time, a partitioned structure did not guarantee a correct interpretation, such that 56% of the partitioned structures were not correctly interpreted. Rearranging the pieces was important to developing a correct interpretation, but children could adapt the environment and still not be prepared to interpret that structure.

### 5.4. Discussion

Children solved more operator problems when they manipulated the pieces than when they drew on a picture of the pieces. It is important to note that in most cases the same child who could do a problem with physical materials could not do a similar problem with a picture.

Different hypotheses could explain the superiority of the physical condition. The PDL hypothesis is that moving the pieces permitted children to gradually adapt their environment and in the process change their interpretations. In the pictorial condition, the children interpreted the pieces as whole numbers, and they chose one and/or four pieces in response to a problem such as ¼ of 8. This was a sensible response, given their many prior experiences counting objects. When taking action is hard, people are more likely to follow a preexisting interpretation (O’Hara & Payne, 1998; Svendsen, 1991). This may be what children did with the pictures. They may have chosen the first interpretation available.

In contrast, moving the pieces in the physical condition helped the children overcome their previously correct but misapplied interpretation. As Shirouzu et al. (2002) argued, a benefit of physical manipulation is that it supports reinterpretation of the visible. This reinterpretation could have happened in the following way: The children naturally collected the pieces into piles and moved them around. The movement of the piles relative to one another, plus a pile’s collection within a single hand, created a gestalt that helped children see the piles as distinct entities. This helped the children let go of their single-piece whole-number interpretation so they could sometimes count a collection of pieces as a single group. Coupled with an intuitive constraint that the groups should be the same size, the children were able to partition the pieces into the number of groups shown in the denominator of the fraction and sometimes appreciate that the number of pieces in one group constituted the answer.

An alternative to the PDL hypothesis is that the children did not learn the correct interpretation by interacting with the situation. Instead, the children already had a grouping schema for fractions that regulated both their adaptations and interpretations, and the physical materials helped them execute that schema by plan. An example of this sort of theorizing comes from the work on speech–gesture mismatch (Goldin-Meadow, Alibali, & Church, 1993). A typical mismatch example occurs with the liquid conservation task where children need to determine whether a tall, narrow glass has the same amount of liquid as a short, wide glass. Young children often say the glasses have the same amount of water because they are at the same height, but with their hands, they gesture indicating the width. The speech–gesture mismatch can foreshadow the development of mature liquid conservation (Church & Goldin-Meadow, 1986). The common explanation for this phenomenon is that children have a single underlying representation that is being expressed in different modalities with differing degrees of effectiveness (McNeil, 1987). By analogy, in the case of the manipulatives, the children might have a single underlying representation that regulates their ability to “express” both the adaptation and its interpretations.

The evidence from the physical condition did not support this alternative hypothesis. First, the children moved the pieces much more than necessary to solve the problems by plan. Second, the level of movement was the same for correct and incorrect answers, which would not occur if the children were following a plan to achieve the correct answers. Finally, the children did not always interpret a partitioned adaptation into countable groups, which suggests they did not know the goal state of their manipulations from the outset.

However, these data primarily came from the physical condition. The alternative hypothesis has its greatest credibility in explaining the failure in the pictorial condition: Children may have had a plan, but its execution was too difficult in the pictorial condition. Children may have performed worse in the pictorial condition because they did not think to use their plan, or because they did not have the cognitive resources to imagine rearranging the pieces into groups. Experiment 2 addressed these possibilities.

## 6. Experiment 2: Effects of Action II

The goal of Experiment 2 was to replicate and extend the findings of Experiment 1. In particular, we wanted additional evidence that physical manipulation helped children develop the fraction interpretation and did not just enable children to execute an already formed interpretation by off-loading some of the burden to the environment (Quadrant 2). As before, all the children solved problems with physical and pictorial materials. However, we introduced three variations. First, the experiment involved younger children who had even less experience with fractions. Second, the experiment compared children's ability to solve the operator problems when described numerically (e.g., ¼ of 8) versus when described as a sharing problem. Finally, children received the pieces randomly arranged or they received the pieces preorganized into appropriate partitions.

We included younger children who had little instruction with fractions and manipulatives to reduce the chances that they had a school-based schema for solving these operator problems. A difficulty with this approach, however, is that children need some prior knowledge to understand what they are being asked to do. To get around this difficulty we asked some of the children to solve the problems in the context of a story about sharing fairly. For example, we asked the children how much one person should get if four people are going to share eight pieces fairly. Children of this age have a good understanding of fair-share contexts, so their everyday knowledge may help constrain their activities (Empson, 1999; Streefland, 1993).

Half of the children heard fair-share problems and the other half heard the original numerical statement of the problem. If the children could not solve the problems for the numerical version (with or without manipulatives), it would show they did not have prior knowledge of a partitioned adaptation solution to the problem. If the physical materials still showed an advantage over the pictorial materials for the fair-share context, it would show that the benefits of physical manipulation in this study were not the result of executing an algorithm children knew.

The remaining change involved the presentation of the pieces. One presentation was as before, with pieces placed randomly on the table or in the picture. The other presentation placed the pieces in configurations that matched partitioned adaptations. Fig. 5 shows an example of

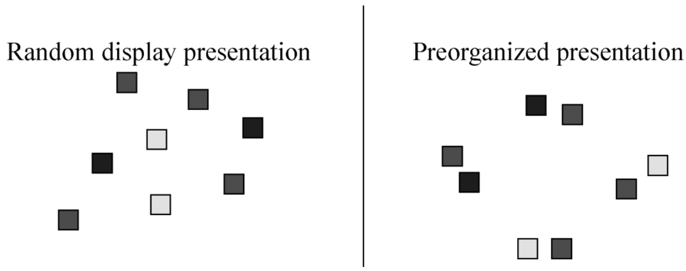


Fig. 5. Random display and preorganized material presentation for ¼ of 8. As in Experiment 1, the random display presentation lays the materials out over the space or table in an unorganized fashion. The preorganized presentation shows the materials partitioned in an effective way for solving the problem.

the two presentations. If the sole benefit of manipulation is alleviating working memory burdens, then the preorganized presentation should show a strong advantage over the random presentation; the preorganized presentation removed the need to imagine the intermediate or final states. In addition, if children had a preexisting schema for these problems, then presumably they would recognize the significance of the preorganization for the solution. Alternatively, if children did not have a solid schema, they might not recognize the significance of the preorganized materials, and the arrangement would not help them solve the problem. If so, this would suggest that children at this stage of knowledge development need the interactive process of gradually adapting their environment and refreshing their interpretations so the partitions can yield interpretations of groups.

### *6.1. Methods*

#### *6.1.1. Participants*

Twenty 9- and 10-year-old children were randomly selected from two new fourth-grade classrooms at the beginning of their school year. The children had not used manipulatives for operator problems, although they had done standard third-grade tasks such as identifying fractions and showing fractional amounts (e.g., shade one-fourth of four pieces).

#### *6.1.2. Design and procedure*

The procedure was similar to Experiment 1 with individual videotaped sessions, no feedback, no time limit, and randomized problem order. The dependent measures were the same as for Experiment 1, and as before, all children solved half of the problems using pictures and half of the problems using physical pieces. New to this experiment, we presented half of the problems with the pieces preorganized and half with the pieces randomly arrayed for both the physical and pictorial conditions. Also new to the experiment was the between-subject factor of problem context. Children in the number context heard the fraction instructions as before (e.g., “Make two-fifths of these 10 pieces”). Children in the story context heard the fair-share scenario (e.g., “Here are 10 pieces. Imagine you had to share these pieces equally with five people. How many pieces would two people get?”).

The coding for interpretation and adaptation was the same as in Experiment 1. It should be noted that children could make nonpartitioned adaptations even when they received the pieces preorganized into groups. For example, to show ¼ of 8 in the pictorial condition, children might ignore the four groups of two pieces and circle one piece. Alternately, they might circle four pieces out of different groups. In the physical condition, children might recombine all the pieces into one big pile as they started working.

### 6.2. Results

Fig. 6 presents the effects of the physical and pictorial conditions on children's interpretations and adaptations by problem context. For the story context, physical manipulation improved children's interpretations, and nearly all the children in the physical and pictorial conditions made partitioned adaptations. Therefore, the benefit of manipulation was not simply to create a final configuration from which children could read off the answer, because both conditions made appropriate final configurations. Manipulating the physical materials also helped children reinterpret the adaptations they created. For the number context, physical manipulation mildly but nonsignificantly increased the frequency of partitioned structures. However, given the impoverished context and their lack of fraction knowledge, children could not interpret the partitioned structures very well and had the same level of interpretation as when they worked with the pictorial materials.

Fig. 7 displays the effects of the random and preorganized presentations of the materials by problem context. For both the story and the number context, children's interpretations exhibited a modest but nonsignificant improvement when they received the preorganized presentation. Surprisingly, the children's final adaptations were not affected by the organization of the materials. Children in the story condition were near ceiling regardless of initial presentation, whereas children in the number condition tended to neglect the preorganized structure and organize the pieces their own way. The children's adaptations were not dependent on the configurations that were handed to them.

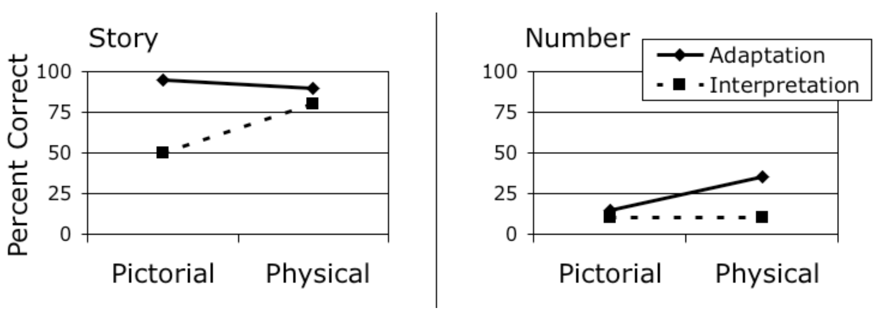


Fig. 6. Effects of material and problem context on interpretation and adaptation, Experiment 2. In Experiment 2, children in the story context made correct interpretations and partitioned adaptations more often than children in the number context with both physical and pictorial materials. Children in the story context made partitioned adaptations as often with pictorial as with physical materials, although they correctly interpreted their adaptations more often when they manipulated the pieces. Children in the number context made partitioned adaptations more often with the physical materials, but this did not improve their interpretations.

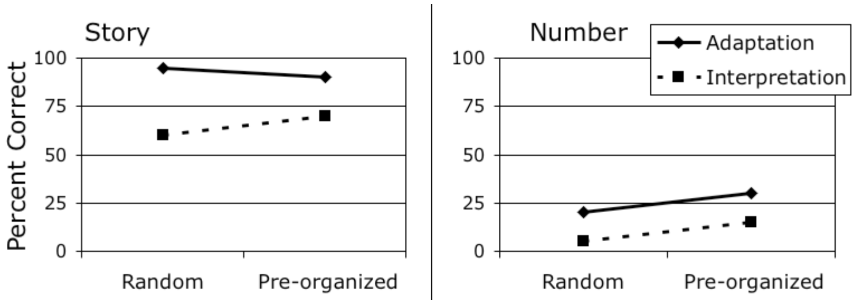


Fig. 7. Effects of display organization and problem context on interpretation and adaptation, Experiment 2. In Experiment 2, children in the story context made correct interpretations and partitioned adaptations more often than children in the number context with both random and organized presentation of the materials. Children in the story context made partitioned adaptations as often with random and organized presentation of the materials, although they correctly interpreted their adaptations slightly more often with the organized presentation of the materials. Children in the number context made partitioned adaptations and interpreted them correctly with similar frequency with both presentations.

### *6.3. Measures taken separately*

We used repeated measures logistic regression models (the SAS procedure GENMOD, version 8.1). The within-subjects factors were material (physical, pictorial) and organization (random display, preorganized display), and the between-subject factor was context (number, story). We first evaluated main effects and then introduced interaction terms to see if they provided any additional explanation of the variance. For the analysis of the interpretations, there was a significant effect of context, $\chi^2\,(1, N = 20) = 12.4, p < .001$, and of material, $\chi^2\,(1, N = 20) = 4.5, p < .05$, but no effect of organization. Adding interaction terms to the model provided no additional explanation of variance. For the adaptation analysis, there was an effect of context on the frequency of partitioned adaptations, $\chi^2\,(1, N = 20) = 15.3, p < .001$, but no effect of material or organization (see Table 3). Adding interaction terms to the model provided no additional explanation of variance.

On the activity measures, a multivariate analysis crossed all the conditions and showed no main or interaction effects on the dependent measures of the number of moves and restarts. Descriptively, the most notable trend was that the children made more moves with the pictures than with the physical pieces, which was the opposite result from Experiment 1. This indicates that the difference between the physical and pictorial materials is not action per se. Additional trends were that children made more moves with the story context, as might be expected given that they had the fair-share story to guide their activity. Children also exhibited a trend to make more moves and restarts with the randomly organized pieces than with the preorganized ones.

#### *6.3.1. Relationship between measures*

Table 4 shows how children's interpretations and their level of activity related to adaptations over time. We reaggregated the data the same way as in Experiment 1.

As in Experiment 1, correct interpretations and partitioned adaptations did not appear to come from a common underlying schema. Correct interpretations were associated with parti-

Table 3
Measures broken out by problem context, material, and organization

| Condition | Correct Interpretations | Partitioned Adaptations | Moves (*SE*) | Restarts (*SE*) |
|---|---|---|---|---|
| Number ($n = 10$) | | | | |
| Physical | | | | |
| Random display | 10% | 30% | 22.2 ( 6.5) | 1.5 (0.3) |
| Preorganized | 10% | 40% | 17.6 ( 3.7) | 1.3 (0.2) |
| Pictorial | | | | |
| Random display | 0% | 10% | 27.4 (12.7) | 1.2 (0.2) |
| Preorganized | 20% | 20% | 22.9 ( 7.9) | 1.1 (0.1) |
| Story ($n = 10$) | | | | |
| Physical | | | | |
| Random display | 80% | 90% | 27.4 ( 9.7) | 1.4 (0.2) |
| Preorganized | 80% | 90% | 22.6 ( 5.5) | 1.3 (0.3) |
| Pictorial | | | | |
| Random display | 40% | 100% | 43.7 (14.5) | 1.3 (0.3) |
| Preorganized | 60% | 90% | 22.8 ( 8.4) | 1.1 (0.1) |

tioned adaptations, but partitioned adaptations did not guarantee correct interpretations. When we collapsed the conditional probabilities across conditions, a partitioned structure yielded a .46 probability of making a correct verbal answer, whereas a nonpartitioned structure yielded a .06 probability of making a correct verbal answer. The chance of making a partitioned structure was .67 across the physical materials and .53 across the pictorial materials. Thus, the physical materials helped the students make more partitioned structures, which presumably improved interpretation. At the same time, a partitioned structured did not guarantee a good solution, such that 54% of the partitioned structures were not correctly interpreted. Thus, whether children received a prepartitioned display or created their own, the partitioned structure was not sufficient for generating the correct interpretation, although it did help.

### *6.4. Discussion*

Again, physical manipulation exhibited an advantage over pencil and paper. This study helped to further show that the benefit of manipulation was due to distributed learning and not simply due to off-loading known operations to the environment. The opportunity to rearrange the pieces helped the children develop a fraction interpretation more than being presented with a useful "off-loading" structure at the outset.

With respect to interpretation, the children who received the fair-share version of the problems had an advantage. Children who received the number version did not benefit interpretively from the opportunity to manipulate the pieces, although they did exhibit a trend to make more partitioned adaptations. The fact that children of this age could not solve the number version of the problem suggests that the advantage of manipulation in the fair-share version was not due to a preexisting fraction algorithm. Moreover, the fair-share version was not sufficient for the children to generate an efficient plan. The children consistently made

Table 4
Interpretations and moves broken out by adaptation over problem-solving tries, Experiment 2

| | Adaptations | Correct[a] | Moves (*SE*) | |
|---|---|---|---|---|
| | | | Correct | Incorrect |
| Story context | | | | |
| Physical materials | | | | |
| Try 1 (*n* = 10) | partitioned 90% | 56% | 11.4 (2.8) | 17.5 (4.0) |
| | ~partitioned 10% | 100% | 1.0 (n/a) | — |
| Try 2 (*n* = 10) | partitioned 90% | 67% | 32.7 (10.4) | 19.0 (0.5) |
| | ~partitioned 10% | 0% | — | 8.0 (n/a) |
| Try 3 (*n* = 5) | partitioned 100% | 60% | 16.3 (9.7) | 16.5 (4.6) |
| | ~partitioned 0% | — | — | — |
| Pictorial materials | | | | |
| Try 1 (*n* = 10) | partitioned 90% | 56% | 30.6 (10.7) | 26.0 (5.5) |
| | ~partitioned 10% | 0% | — | 19.0 (n/a) |
| Try 2 (*n* = 10) | partitioned 90% | 44% | 29.5 (17.8) | 34.2 (11.9) |
| | ~partitioned 10% | 0% | — | |
| Number context | | | | |
| Physical materials | | | | |
| Try 1 (*n* = 10) | partitioned 40% | 25% | 5.0 (n/a) | 16.3 (5.6) |
| | ~partitioned 60% | 0% | — | 13.0 (5.2) |
| Try 2 (*n* = 10) | partitioned 40% | 25% | 19.0 (n/a) | 22.3 (1.2) |
| | ~partitioned 60% | 0% | — | 16.3 (4.9) |
| Try 3 (*n* = 4) | partitioned 50% | 0% | — | 18.5 (3.2) |
| | ~partitioned 50% | 0% | — | 16.0 (8.5) |
| Pictorial materials | | | | |
| Try 1 (*n* = 10) | partitioned 10% | 0% | — | 140.0 (n/a) |
| | ~partitioned 90% | 11% | 6.0 (n/a) | 12.1 (1.9) |
| Try 2 (*n* = 10) | partitioned 20% | 0% | — | 29.5 (3.9) |
| | ~partitioned 80% | 0% | — | 18.9 (5.1) |

[a]Percentage of correct interpretations within partitioned/nonpartitioned adaptations.

many more moves than necessary, and the number of moves for the fair-share version was no less than for the number version, where the children had no plan. Instead of providing a plan of action and interpretation, the fair-share story provided a frame for organizing activity, which eventually led to a solution. For example, sharing fairly constrained the children to partition the pieces into equal sets, which children gradually came to interpret as groups.

In the story context, the opportunity to manipulate the pieces was more valuable than showing a preorganized configuration from which the children could potentially "read off" the solution. When allowed to manipulate the pieces, the children showed little benefit from seeing the pieces preorganized. When working with the pictures, the preorganization helped somewhat, but it still did not bring the exact same children to the level they achieved when given the opportunity to manipulate the pieces. Adults, who understand fractions reasonably well, can quickly interpret the significance of the preorganized version of the problem. However, for children, the fraction interpretation of the physical partitions is not immediate, which is one reason that it is a mistake to assume that children can easily associate a physical referent to its symbolic equivalent in educational settings (Martin, 2003; Uttal et al., 1997). Children do not always know which aspect of the referent is relevant (Scaife & Rogers, 1996). However, why would moving the pieces have more sway for developing the interpretation of groups than receiving the partitioned structure at the outset?

One answer is that the opportunity to move the pieces helped the children let go of their initial and well-practiced whole-number interpretation of the pieces. This whole-number interpretation was so strong that, without action, it inhibited the discovery of the grouping interpretation even when the pieces were displayed preorganized. In contrast, the physical manipulation helped the children let go of their initial interpretations. For example, Schwartz and Holton (2000) found that taking physical action, even with closed eyes and without direct contact, can help people let go of a given visual perspective and imagine an object from another point of view. For these children, by interacting with the environment in an open-ended manner, they more easily explored or searched for other possible interpretations and structures.

A second, complementary answer is that the children did not know what problem the partitioned structure solved unless they actively tried to organize the pieces themselves (Bruner, 1973; Vygotsky, 1978). When studying middle school children learning statistics, Schwartz and Martin (2004) found that the active effort to invent ways to solve problems made children much more ready to appreciate a solution when it appears. Simply telling or showing the solution does not help if people do not recognize the problems the solution resolves (Bransford, Franks, Vye, & Sherwood, 1989). By grappling with ways to organize the pieces, for example, to find a way to use all the pieces, children can interpret the significance of groups when their actions happen to place all the pieces into equivalent partitions.

Experiment 2 provides a useful qualification on the value of manipulation. Based on Experiment 1, one might argue that children simply made more movements with the physical materials, which increased their chances of generating the partitioned adaptation. However, in Experiment 2, children made more moves with the pictures than with the manipulatives. This shows that the difference is not activity per se, but rather the types of activities that the environment supports. For the pictures, children could take many actions (e.g., covering pieces with a hand), but they could not easily grab and move piles of pieces to suggest the idea of counting groups as opposed to discrete pieces. This points to the significance of how different physical environments afford different types of actions, which may be more or less useful.

In the first two experiments, we primarily focused on manipulation versus no manipulation. In the remaining experiments, we examined the interaction between different physical environments and manipulation.

## 7. Experiment 3: Effects of knowledge

The preceding experiments demonstrated a PDL "sweet spot" where children can manipulate the environment to help generate new interpretations of quantity. An important goal of these studies was to show that the children were actually learning through manipulation rather than using the environment to help them off-load some of the cognitive burden of implementing a known problem-solving routine. For the next experiment, we wanted to further highlight the uniqueness of PDL by showing behavior on either side of the sweet spot. At one extreme, when people have a great deal of knowledge and do not need to depend on an emergent interpretation, they can impose a preexisting interpretation to repurpose an environment that might not be ideally structured. A person who is good at counting, for example, can count just about any object, and if necessary, even make a continuous quantity into countable discrete units (e.g., piles of sand). At the other extreme, when people have very little knowledge, they may be more prone to environmental structures in their adaptations, and they may not be able to reinterpret the situation when a useful adaptation appears. The structure of the environment shapes behavior but does not yield learning.

The third experiment explored the interaction of prior knowledge and environmental structure with fifth graders who solved multiplication problems and fraction-addition problems. Multiplication served as the high-knowledge task. Children in fifth grade are generally facile with concepts of multiplication, although they often cannot do complex multiplication problems in their heads without pencil and paper. Given a physical situation, they need to adapt the environment to help solve the problems. Therefore, the multiplication problems allowed us to evaluate physically distributed cognition when children have high knowledge.

Fraction addition served as the low-knowledge task, because fifth graders typically have had minimal experience with it in school. Fraction addition is more complex than the operator problems of the first two studies. For the earlier problems, the challenge was to create partitions and interpret them as groups. Fraction addition requires making and interpreting wholes as well as groups. To take a simple case, consider ¼ + 2/4 (harder cases have unlike denominators). Imagine that children have successfully created the two fractions by pulling one piece from a collection of four pieces (¼) and two pieces from another collection of four (2/4). To add them, the children combine the pulled pieces (2 + 1). The challenge is that when they combine the pulled pieces they have to appreciate that they make three out of four, rather than just three pieces. The children need to interpret the pieces as part of a whole that may not be visually apparent

## 7.1. Methods

For both the multiplication and fraction problems, children solved several problems using four different materials. One class of material came as individual units of the same size (tiles and beans). The other class of material had a built-in partitioned structure. One partitioned material was pie pieces, which lend themselves to making a whole with equal subparts. The other was a *geoboard,* a board with a grid of nails and rubber bands that can be used to mark out groups. For the multiplication problems, our hypothesis was that the high-knowledge task of multiplication would enable the children to distribute their cognition to all the physical environments equally well. However, for fraction addition, the children would be more prone to the structure of the environment, such that the partitioned materials would support more partitioned adaptations of the problem. For example, in Fig. 3, three tile pieces and three pie pieces were set apart from the rest. When looking at the tile pieces, it is hard to know they should form part of a whole. In contrast, the pie wedges fit into a circle, so they are more readily viewed as a part of a whole. So, for example, given pies, children would be likely to create a circular "whole" arrangement instead of simply setting the wedges haphazardly next to each other as they might do with tiles. Therefore, we expected the partitioned materials to help children make more partitioned adaptations than the unit materials. Even so, given the prior results that children can adapt environments well but not interpret them correctly, we did not think the children would give many correct verbal answers for the low-knowledge task.

#### *7.1.1. Participants*

Sixteen 10- and 11-year-old children were selected randomly from three fifth-grade classrooms. They had studied multiplication but had minimal exposure to fraction addition. As a manipulation check for our characterization of high- and low-knowledge tasks for this population, the children completed a short preassessment in which they solved single-digit multiplication problems and unlike-denominator fraction-addition problems. For multiplication, 93% of the children could solve a single-digit multiplication problem in their head and on paper. This high rate of success indicated that children indeed had high knowledge of multiplication. In contrast, 13% of the children could do a fraction-addition problem in their head, and 44% could do it on paper. This much lower rate of success indicated that children's knowledge of fraction addition was low. These results matched our expectations based on their standardized mathematics curricula.

#### *7.1.2. Materials*

Children manipulated four different materials: tiles, beans, geoboard, and pies. Tiles and beans are unit materials. Beans came with small paper cups (for grouping) that children could use if they wished. The geoboard and pies are partitioned materials. The geoboard is a 7-in. × 7-in. board with 25 evenly spaced nails and rubber bands for demarcating groups of nails. The students were familiar with these materials. We asked which of the materials they had used in their mathematics lessons, and all students reported that they had used all of the materials.

#### *7.1.3. Design and procedure*

In a fully within-subjects design, the factor of high- and low-knowledge task (multiplication vs. fraction addition) was crossed with the factor of material structure (partitioned vs. unit) with two instances of each material structure. Children sat in front of mounds of the four manipulatives. They chose which manipulative they wanted to use for each problem with the restriction that they could not use a manipulative twice for a problem type. They grabbed whatever pieces they thought were necessary and could return or take more pieces during a problem if they wished. Half the children completed the four multiplication problems first, and half completed the four fraction problems first. For the multiplication problems, we ensured the students needed to use the materials to solve the problems. We found the level of multiplication problem that they could not do mentally and then used that class of problem for the recorded trials (either one-digit multiplicands or two- by one-digit multiplicands). All children solved fraction-addition problems that used denominators that were multiples of one another (e.g., $\frac{1}{4} + \frac{1}{8}$). Children were videotaped in individual interviews. Children had no time limit and did not receive feedback. Dependent measures included interpretations and final adaptations.

#### *7.1.4. Coding*

As before, a correct interpretation was a numerically accurate verbal answer. The coding for adaptations differed slightly from earlier experiments because of the new types of problems.

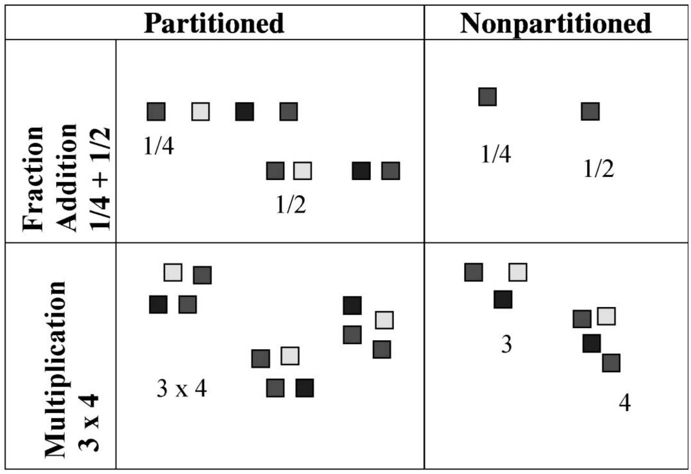


Fig. 8. Adaptations for fraction addition and multiplication, Experiment 3. A partitioned adaptation for ¼ + ½ shows partitioned representations of the two fractions. A nonpartitioned adaptation for fraction addition might show ¼ as one piece and ½ as one piece. A partitioned adaptation for 3 × 4 shows three groups of four pieces each. A nonpartitioned adaptation might show direct representations of the numbers in the problem (e.g., 3 and 4).

Fig. 8 provides a schematic of different adaptations. For partitioned adaptations, children had to collect pieces separately for each fraction and then partition each into groups. For nonpartitioned adaptations, children often simply indicated the number of pieces that corresponded to one of the numerals in each fraction. For multiplication, a partitioned adaptation needed to show a total amount split into groups. For nonpartitioned adaptations, children might simply show the quantity for each number in the problem or the total number of pieces in the answer without any grouping.

### *7.2. Results*

High knowledge enabled the children to use the materials to reach correct answers, whereas low knowledge did not. For each pair of problems within a condition (e.g., unit materials for multiplication), a child could receive a maximum score of 2 points for correct interpretations and 2 points for partitioned adaptations. Table 5 presents the results. The structure of the materials did not affect children's interpretations in either condition, and children were much more accurate for multiplication than for fraction addition. We analyzed the data using a 2 × 2 analysis of variance (ANOVA) on children's interpretations with task and material structure as within-subjects factors. Three children were excluded because they did not receive one of the problems. Analysis revealed a main effect of task, $F(1, 12) = 30.5$, $MSE = .39$, $p < .001$, with children being more successful at multiplication. We found no effect of material structure or interaction between task and structure on interpretations.

Although the materials did not affect the children's interpretation, they did affect the adaptations children made in the low-knowledge condition. For the fraction-addition problems, children made more partitioned adaptations with the partitioned materials than with the unit mate-

Table 5
Average number of good interpretations and adaptations (out of 2 possible) broken out by problem type and material

| Knowledge/Materials | Correct Interpretations (*SE*) | Partitioned Adaptations (*SE*) |
|---|---|---|
| High Knowledge (Multiply) | | |
| Partitioned materials | 1.2 (0.1) | 2.0 (0.0) |
| Unit materials | 1.4 (0.2) | 2.0 (0.0) |
| Low knowledge (add fractions) | | |
| Partitioned materials | 0.4 (0.2) | 0.6 (0.2) |
| Unit materials | 0.4 (0.2) | 0.2 (0.2) |

rials. We analyzed the data using a 2 × 2 ANOVA on adaptation score with the within-subjects factors of task and structure. Two additional children were excluded from this analysis because they did not use the materials to solve at least one problem (they had given up trying). The analysis yielded a main effect for task, $F(1, 10) = 83.1$, $MSE = 0.4$, $p < .001$; a main effect for structure, $F(1, 10) = 5.7$, $MSE = 0.06$, $p < .05$; and an interaction between task and structure, $F(1, 10) = 5.7$, $MSE = 0.06$, $p < .05$. The interaction was due to the effect of the materials' structure in the fraction-addition condition. Simple effects analyses indicated there was no significant difference between the unit and partitioned materials for multiplication. However, for fraction addition, children made better adaptations with partitioned materials, $F(1, 10) = 5.7$, $p < .05$. Thus, with high knowledge, children flexibly adapted all the materials, whereas with low knowledge, they were more prone to the environment and only created useful adaptations when the environment suggested them. However, those adaptations did not immediately translate into more successful problem solving, as shown by the similar interpretation across materials.

### *7.3. Discussion*

Experiment 3 helped locate where physically distributed cognition is most productive. When people already have a strong understanding, as in this case of multiplication, they can repurpose many environments to help with extensive problem solving. The ideas they begin with help them decide how to interpret and manipulate the environment. Although the children could not solve the problems mentally, they were quite successful using the materials to support multiplication, regardless of the material's physical structure. In contrast, when people have very low knowledge, as was the case for fraction addition, the structure of the environment can help shape their activities, but people may not be prepared to reinterpret and capitalize on the resulting environmental structures. In this experiment, children adapted the partition-structured environment in potentially more useful ways than the unit-structured environment, but they were not able to reinterpret these structures to solve problems more successfully. The result was similar to Experiment 2, where children received preorganized arrays of pieces but could not uniformly see the grouping structure.

Finally, similar to Vygotsky's (1978) Zone of Proximal Development, Experiments 1 and 2 documented a window in which hands-on activity can be particularly helpful. When people have incipient understanding, as in the case of the operator problems of Experiments 1 and 2, manipulation can support the emergence of new interpretations that capitalize on new organizations of the physical environment.

## 8. Experiment 4: Effects of materials

Thus far, it is not clear how the structure of the environment affects the narrow window of PDL. The fraction-addition condition of Experiment 3 suggested that the structure of the environment should have an effect on the adaptations available to children with incipient knowledge. Yet, because the children had such low knowledge of fraction addition, the study did not demonstrate any effects of material structure on their interpretations. Thus, the following two experiments examine the effects of material structure on interpretation more closely. To do this, it is particularly important to examine transfer to new settings, because the quality of people's interpretations may be masked by the structure of the initial learning environment. For example, one might imagine a highly engineered environment that nearly guarantees good performance during initial learning. Yet, the environment may do so much of the work that it is unnecessary for people to develop an interpretation of how the environment enables this work. Cash registers that calculate correct change for store clerks do not necessarily help those clerks to do math when they can no longer rely on the cash register. Therefore, performance in an initial learning environment may be a bad index of whether people have developed abstract mathematical ideas. A transfer task that requires individuals to operate in a new material environment can be a better index of understanding.

We made a tentative hypothesis about the types of materials that are likely to support PDL and its extension to new environments at transfer. We hypothesized that a supportive physical environment for learning includes materials that require (and enable) children to generate new structures and interpretations that are central to a concept. This will help them understand the cognitive function that a structural form needs to support, and this will enable them to repurpose new situations to recreate this functionality. In the following experiments, we ask children to learn fraction addition. One central conceptual move for fraction addition is to view added pieces as part of a new whole (e.g., $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and not just 2).

We also hypothesized that an unsupportive environment for learning includes physical materials that obviate the need for adaptation and reinterpretation around a central concept. The environment may support the induction of how to solve problems in that specific environment, but one consequence of an overengineered environment is minimal transfer to new environments (Gilmore, 1996). The children will have difficulty when they confront situations that do not include equal structure. They will have only induced efficient procedures for off-loading cognitive operations in a specific task environment.

To test these tandem hypotheses, children learned with either pies or tiles, and then we assessed how well they transferred to new materials. We thought that children who learned with pies would transfer more poorly. Learning with pie pieces shortcuts the need for adaptation and reinterpretation by providing the whole for free. For example, with pies, children can solve a problem such as $\frac{1}{3} + \frac{1}{3}$ by pushing two $\frac{1}{3}$ wedges together. They may never explicitly notice how the circle is making the two wedges appear as $\frac{2}{3}$ of a whole instead of just 2 pieces. Although the children can solve $\frac{1}{3} + \frac{1}{3}$, the hidden consequence of the pies is that children may never grapple with the adaptations needed to spawn new interpretations involving wholes. If our analysis is correct, it would help explain why children have trouble developing the concept of a whole when working with pie pieces (Baroody & Lai, 2002) and why some researchers have found pies ineffective for instruction (Kerslake, 1986; Mack, 1990; Moss & Case, 1999). The pies do too much work. In contrast to the pies, the tiles require children to adapt the environment so that they can explicitly interpret the presence of a whole. For example, three tiles by themselves do not carry the idea of three tiles out of a larger specific whole. The children have to adapt the environment and their interpretations so the tiles can carry this function. We thought that this process of adapting and reinterpreting the tile pieces would be better for learning as measured by children's ability to work with new environments at transfer.

Children learned over three sessions to solve increasingly difficult fraction-addition problems with either pies or tiles. At the end of each session was a transfer phase in which children tried to solve problems with different materials. In the transfer phase, children worked on problems at the same level of difficulty as the problems they had worked with earlier in the session. Our hypothesis was that the tile children would do better at transfer than the pie children would, because they would be less dependent on the structure of the initial learning environment and would be able to adapt and reinterpret the new materials.

### *8.1. Methods*

#### *8.1.1. Participants*

Sixteen 10- and 11-year-old children from three fifth-grade classrooms were assigned randomly to the pie and tile conditions, with the restriction that the children's achievement levels (based on state scores) were similar across conditions. Equating achievement levels was important because initial differences could have amplified over the course of the learning phase, which we wanted to prevent.

#### *8.1.2. Materials*

The study used tiles, pies, bars, and beans without cups. Bars are plastic rectangles that are divided into halves, thirds, and so on. Bars and pies are similar in that they are both divided into fractional pieces and use the same color scheme (e.g., the halves in both sets are pink). Beans and tiles are relatively similar because the pieces come in one size.

#### *8.1.3. Design*

A between-subject factor determined whether children's base learning materials were tiles or pies. In a microgenetic design over 3 days, the children learned to complete progressively complex fraction-addition problems with their base material (see Table 6). On each day, children completed two phases: (a) a learning phase with their base material that included informative feedback, and (b) a transfer phase using new materials without feedback. For the learning phase, the primary dependent measures were the children's highest problem level mastered each day, the level of feedback children needed to succeed at each level, and the final adapta-

Table 6
Problem levels, Experiment 4

| Sum | Same Denominator | Multiples Denominators | Different Denominators |
|---|---|---|---|
| Sum < 1 | Level 1 (e.g., 2/4 + 1/4) | Level 3 (e.g., 1/12 + 5/6) | Level 5 (e.g., 1/4 + 1/3) |
| Sum > 1 | Level 2 (e.g., 2/3 + 2/3) | Level 4 (e.g., 2/4 + 5/8) | Level 6 (e.g., 2/3 + 3/4) |

tion used on each problem. For the transfer phase, children received each level of problem they had tried during learning. They used three transfer materials for each problem level—bars, beans, and the base material used by children in the other condition (pie children used tiles at transfer, and tile children used pies). The dependent measures at transfer were interpretations and final adaptations.

#### *8.1.4. Procedure*

The children worked with the researcher one-on-one in three videotaped sessions spread over a week. On each day in the learning phase, children began at Level 1 and progressed as far into the problem levels (Table 6) as they could within 20 min. The researcher first asked the children to solve the problem in their head. Afterward, the researcher asked the children to solve a Level 1 problem using their base material. If the children succeeded on two problems in a row without feedback support, the researcher repeated the process with the next level of problem. If the children made a mistake, the researcher provided instructional feedback (described in the following paragraph) until the children could solve two new problems in a row without further feedback. During the 10-min transfer phase of each day, the children tried to solve problems with each transfer material at all the levels they had attempted that day.

As mentioned, when children did not solve a problem correctly in the learning phase, they received increasingly informative feedback. The first time they were incorrect at a problem level, they received Level 1 feedback. If they were incorrect again, they received Level 2 feedback, and so on. There were four levels of feedback support as shown in Fig. 9. At Level 1 we showed children what the end state of the problem would look like with their materials and told them the answer. At Level 2 we repeated the first hint and then showed children how to set up the partitions for the problem. At Level 3 we repeated Levels 1 and 2 and then demonstrated how to make particular fractions with the materials. At Level 4 we repeated the previous levels and showed children how to combine the two fractions that they had made with the materials. The grouping method shown in Fig. 9 works for both tiles and pies.

### *8.2. Results*

#### *8.2.1. Initial learning with base material*

Children in the pie and tile conditions ultimately learned to the same level of verbal accuracy with their base materials. Students received a score of 1 to 6 each day for the highest level problem they mastered that day (two problems correct in a row). Table 7 shows the highest average problem level the children achieved each day. In a 2 × 3 ANOVA, learning condition was a between-subject factor and day was a within-subjects factor. Children in both groups signifi-

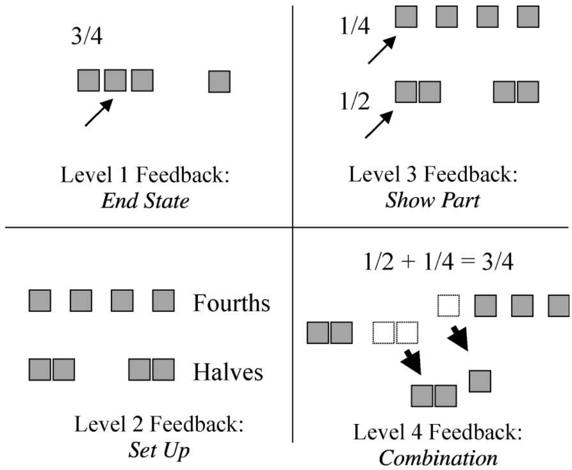


Fig. 9. Feedback levels, Experiment 4. For example, for the problem ¼ + ½, Level 1 showed children the end state and stated, "The answer is ¾." Level 2 showed children how to setup the fractions in the problem, "Use four pieces to make both halves and fourths." Level 3 indicated what part showed the particular fraction, "Here's one whole, and here's one fourth of it. Here's another whole, and here's one half of it." Level 4 modeled the addition process, "Put the ¼ and the ½ together to get ¾."

cantly improved over the course of 3 days, $F(2, 28) = 21.5$, $MSE = 0.5$, $p < .001$. The analysis revealed no significant effect of learning condition or interaction between learning condition and day. Descriptively, the pie group did not do as well as the tile group on the 1st and 2nd days, but they caught up by the 3rd day.

Table 8 indicates that the children in both conditions did not differ in the amount of instructional feedback they needed. Interestingly, Level 1 feedback, which simply showed the end state, was sufficient about half of the time for the children to infer the intermediate steps needed to reach the end state. Novick and Morse (2000) found that the ease of seeing interme-

Table 7
Highest problem level passed, learning phase, Experiment 4

| Group | Day 1 (*SE*) | Day 2 (*SE*) | Day 3 (*SE*) |
|---|---|---|---|
| Tiles | 1.63 (.25) | 2.75 (.29) | 3.00 (.43) |
| Pies | 0.88 (.28) | 1.75 (.39) | 2.75 (.23) |

Table 8
Level of instructional feedback as a percentage of total feedback, Experiment 4

| Group | Level 1 | Level 2 | Level 3 | Level 4 |
|---|---|---|---|---|
| Tiles (%) | 47 | 22 | 25 | 6 |
| Pies (%) | 55 | 21 | 21 | 3 |

diate steps in a final-state diagram of an origami figure predicted people's subsequent ability to recreate the figure. In this study, children who had tried but failed to structure the materials were evidently in a good position to extract the intermediate states. This is a useful educational finding, because it shows that under appropriate conditions, teachers can have children infer a mathematical procedure rather than telling them a procedure they might just copy without comprehension.

### *8.2.2. Transfer to new materials*

In the transfer phase, children only tried problems at the levels they had attempted that day. To equate children who had reached different levels, we computed percentage correct across problems and materials. We computed two different transfer scores. A problem solving transfer score measured how well children did with the new materials for problems they showed they could solve mentally during the learning phase. We called this *problem solving transfer* because the children already had a mental schema for the problems that they could apply. The learning-at-transfer score measured how well children did with the new materials on problems they could not do mentally during the learning phase. We called this *learning at transfer* because the children had to learn how to rely on the transfer materials to solve the problems.

Fig. 10 shows that the tile children did better than the pie children on both forms of transfer, with a descriptively greater advantage for the learning-at-transfer measure. The drop-off for both groups on Day 3 reflected the fact that the children were working on harder problems. (Recall that children successfully solved harder problems each day in the learning phase and, therefore, received harder problems in the transfer phase.)

Collapsing data across the days, the tile children were correct on 95% ($SE$ = 2%) of the problem-solving transfer problems and 55% ($SE$ = 6%) of the learning-at-transfer problems. The corresponding numbers for the pie children were 89% ($SE$ = 3%) and 35% ($SE$ = 4%). A 2

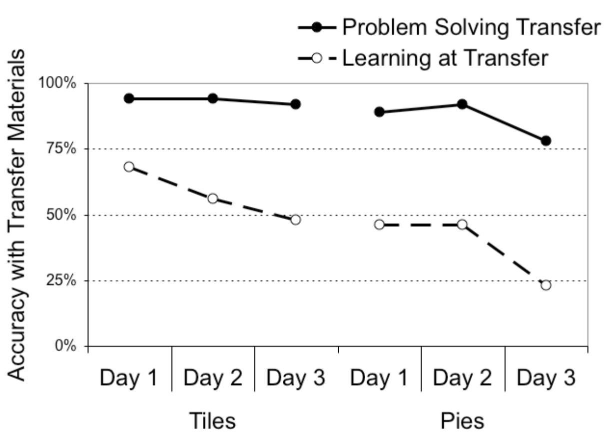


Fig. 10. Transfer by condition, day, and ability to solve problems mentally, Experiment 4. Problem-solving transfer problems are those that children could solve without materials. Learning-at-transfer problems are those that children had not yet successfully solved without materials. Both the pie and tile groups performed better at transfer on problem-solving problems. The pie group performed worse than the tile group on learning-at-transfer problems.

× 2 ANOVA used transfer type (problem solving vs. learning) as the within-subjects factor and learning condition (tiles, pies) as the between-subject factor, with percentage of correct solutions across the materials as the dependent measure. Analysis revealed a main effect of learning condition on accuracy, $F(1, 14) = 12.0$, $MSE = 0.01$, $p < .01$. Children who learned to solve problems with tiles solved similar problems with new materials more frequently than children who learned to solve the problems with pies. We also found a main effect of transfer type, $F(1, 14) = 107.1$, $MSE = 0.02$, $p < .001$. As in Experiment 3, when children had high knowledge as indicated by their ability to solve problems mentally, they were better able to use many materials. The descriptively apparent interaction in Fig. 10 between transfer type and learning condition was not significant, $F(1, 14) = 2.7$, $MSE = 0.02$, $p = .12$.

The tile children also showed superior adaptations at transfer. With the transfer materials, the tile children created partitioned adaptations 90% of the time compared to 59% of the time for the pie children. Table 9 shows the proportion of partitioned adaptations (and interpretations) for each material for each day (collapsed across problem solving and learning transfer). A 2 × 3 × 3 ANOVA used learning condition (pies, tiles) as the between-subject factor with transfer material (bars, opposite, beans) and day as within-subjects factors. Proportion of partitioned adaptations was the dependent measure. Analysis revealed a main effect of learning condition, $F(1, 14) = 7.4$, $MSE = 0.48$, $p < .05$, and a main effect of transfer material due to a small advantage for the bean materials, $F(2, 28) = 3.4$, $MSE = 0.05$, $p < .05$. We found no other effects; *F*s < 1.2. Tile children made more partitioned adaptations regardless of day or material. Notably, the tile group did better with the bar materials than the pie children did, even though the bars had a structure and color scheme that is more similar to the pies.

Table 9
Proportion of good interpretations and adaptations at transfer by condition and material, Experiment 4

| | Correct Interpretations | | Partioned Adaptations | |
|---|---|---|---|---|
| Day/Material | Tiles (*SE*) | Pies (*SE*) | Tiles (*SE*) | Pies (*SE*) |
| Day 1 | | | | |
| Bars | .8 (.1) | .5 (.1) | .8 (.1) | .5 (.1) |
| Other | .8 (.1) | .7 (.2) | .8 (.1) | .3 (.1) |
| Beans | .8 (.1) | .7 (.1) | .9 (.2) | .6 (.2) |
| *M* | .80 | .63 | .83 | .47 |
| Day 2 | | | | |
| Bars | .8 (.1) | .8 (.1) | .9 (.1) | .7 (.1) |
| Other | .6 (.0) | .8 (.1) | .9 (.1) | .6 (.1) |
| Beans | .7 (.0) | .8 (.1) | 1.0(.1) | .8 (.1) |
| *M* | .70 | .80 | .93 | .70 |
| Day 3 | | | | |
| Bars | .7 (.1) | .6 (.1) | .9 (.1) | .7 (.1) |
| Other | .7 (.1) | .5 (.1) | .9 (.1) | .5 (.1) |
| Beans | .8 (.1) | .5 (.1) | 1.0 (.1) | .5 (.1) |
| *M* | .73 | .53 | .93 | .57 |

### 8.3. Discussion

All children improved with their primary learning material, but those who learned with tiles transferred to new materials more effectively; they generated better adaptations and interpretations of the new materials. In contrast, the pie children were dependent on the pie structure to such a degree that they even did worse than the tile children with the colored bars, which are similar to pies. Our explanation is that the pie children did not have to learn to adapt materials to support the reinterpretation of parts and wholes because they got parts and wholes "for free." The tile children had to discover how to adapt and interpret wholes with their unit materials, and this helped when confronting new materials. These results point to an important difference between interface design and learning design. Interfaces that make tasks easier for problem solving may not be the best interfaces for learning (Gilmore, 1996; O'Hara & Payne, 1998; Svendsen, 1991).

One alternative explanation for these results is that the tile group learned a more general procedure for solving the problems. A problem with this explanation is that the two groups learned the same procedure for solving fraction problems. We did not teach the pie children a narrow procedure that could only conceivably work for pies, because this would make the failure at transfer less interesting. Thus, the difference was not in the instructive feedback they received but rather in the materials they manipulated. However, one might argue that we taught the pie and tile children a counting and grouping method that favored the tile condition because it did not capitalize on the unique geometric properties of pies. The pie children may have seen the wholes and parts demonstrated by the pies and been confused by the counting method we taught them. Consequently, the instruction may have had some negative consequences for the pie children's overall understanding. Experiment 5 directly addresses this issue by teaching the pie children a "pie-friendly" method and the tile children a "tile-friendly" method.

## 9. Experiment 5: Effects of materials II

Experiment 5 tested whether the pie children were disadvantaged by the teaching method in Experiment 4. The instructional method did not explicitly take advantage of the spatial properties of the pies, which might account for the advantage of using the tiles. To test this question, the pie group learned a method that used the spatial equivalence of the pies (a $\frac{1}{4}$ piece is the same size as two $\frac{1}{8}$ pieces). Our hypothesis was that instruction in using this method, which capitalizes on the pie's spatial structure (see Fig. 11), would still not put the pie children ahead of the tile children in transfer situations. The method depends on the specialized structure of pies, and children would not develop "stand-alone" ideas that could operate independently of the specialized environment.

### 9.1. Methods

#### 9.1.1. Participants

Nine- and 10-year-old children (fourth grade) were assigned randomly to one of two rooms. Each room implemented a different instructional treatment, pie or tile instruction. Sixteen children were selected randomly from each condition to participate in the interview.

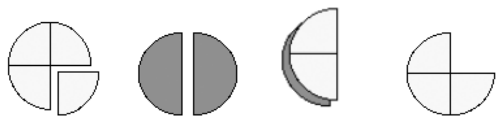


Fig. 11. Pie trading method for ¼ + ½. In Step 1, we showed how to find the pieces that fit four into one whole circle and take out one of them to make ¼. In Step 2, we showed students how to find the pieces that fit two into one whole circle and take out one of them to make ½. In Step 3, we showed students how to fit fourths into the half to find out how many of them to trade. Finally, in Step 4, we showed students how to put all the fourths together to get ¾.

### *9.1.2. Design and Procedure*

For 3 days, the children worked as a whole class on same- and different-denominator fraction-addition problems with either tiles or pies. On the 4th day, in individual videotaped interviews, children solved fraction-addition problems with pies and tiles interspersed with other problems.

In the tile condition, the children learned the “counting and grouping method” exemplified by the Level 4 feedback of Experiment 4 (see Fig. 9). In the pie condition, children learned a “trading method” that relied on the proportional sizes of the pieces (Fig. 11 shows the trading method for adding ¼ + ½). The steps were as follows: (a) Pull a one-fourth piece from the whole set, (b) pull a one-half piece from the whole set, (c) trade the one-half piece for two one-fourth pieces, and (d) join and count the fourths pieces to come up with three fourths.

## *9.2. Results*

As before, learning with tiles had a greater positive effect on interpretations and adaptations when transferring to new materials. Children solved two pie and two tile problems. Table 10 shows the average scores. With respect to the accuracy of their interpretations, tile children scored the same with tiles and pies, whereas the pie children performed better with pies but worse with tiles. A 2 × 2 ANOVA used learning condition (tiles or pies) as a between-subject

Table 10
Proportion of good interpretations and adaptations at transfer by condition and material, Experiment 5

| | Correct Interpretations (*SE*) | Partitioned Adaptations (*SE*) |
|---|---|---|
| Learned With Tiles | | |
| Familiar materials (tiles) | 1.1 (0.1) | 1.6 (0.2) |
| Unfamiliar materials (pies) | 1.1 (0.1) | 1.2 (0.2) |
| Learned with pies | | |
| Familiar materials (pies) | 1.5 (0.1) | 2.0 (0.0) |
| Unfamiliar materials (tiles) | 0.9 (0.1) | 0.6 (0.2) |

factor and familiarity of the materials (base or transfer material) as the within-subjects factor. Analysis revealed a main effect of familiarity, $F(1, 30) = 6.3$, $MSE = 0.3$, $p < .05$, and an interaction between familiarity and learning condition, $F(1, 30) = 6.3$, $MSE = 0.3$, $p < .05$. Simple main effects revealed that on problems using their base materials, the pie children performed better than the tile children, $F(1, 30) = 5.6$, $MSE = 0.3$, $p < .05$, whereas with transfer materials, there was no difference between the conditions. However, the pie children performed worse with the transfer material than with their base material, $F(1, 30) = 13.4$, $p < .01$, whereas the tile children's performance showed no difference. Thus, pie children were better with their base material, and the tile children were more able to learn to solve the problems with the transfer material.

With respect to adaptations, the tile children performed similarly with both materials. In contrast, the pie children correctly partitioned all the pie problems but did poorly with the tiles. Learning condition was the between-subject factor, and familiarity of the materials was the within-subjects factor in a 2 × 2 ANOVA. Analysis revealed a main effect of familiarity on the number of partitioned adaptations, $F(1, 30) = 57.2$, $MSE = 0.2$, $p < .001$. Children made more partitioned adaptations with their base material. We also found an interaction between learning condition and familiarity, $F(1, 30) = 17.8$, $MSE = 0.2$, $p < .001$. Simple main effects revealed that the tile children performed worse than the pie children with their base material, $F(1, 30) = 6.8$, $MSE = 0.2$, $p < .05$, whereas the pie children performed worse than the tile children with the transfer materials, $F(1, 30) = 6.2$, $MSE = 0.5$, $p < .05$. Thus, although the pie children started higher with their base material, they adapted the transfer materials worse than the tile children did.

### *9.3. Discussion*

The advantage of the tile children at transfer suggests that the results of Experiment 4 were not due to a mismatch between the teaching method and the pie materials. Children who learned with the grouping and counting methods with the tiles were equally accurate with pies at transfer and exhibited a small drop in partitioned adaptations when working with pies. In contrast, the pie children learned to use the pies quite well with the trading method of instruction but exhibited a large drop in interpretation and adaptation when transferring to the tiles. The trading method depended on the built-in spatial structure of the pies, and the children did not develop an interpretation of fractions or an ability to take actions that could extend beyond the pies.

An alternative account of the results is that pies are simply easier than tiles. The tile students did not drop on the pie problems because pies are easy. The pie students dropped on the tile problems because tiles are hard, and the tile students had learned a specific method to help them. The problem with this account is that the tile students did not actually use the "easy" geometric structure of the pies during transfer. They treated the pies as though they were tiles, for example, by making groups with pie wedges of different sizes (e.g., putting a 1/3 and 1/6 piece together to indicate two pieces). Therefore, the tile children had learned a method of adaptation and interpretation that interfered with their ability to interpret the useful structure of the pies, but it was still general enough to let them adapt any material successfully.

## 10. General discussion

### *10.1. Empirical summary*

Five studies examined whether, when, and how 9- and 10-year-old children learn by distributing their cognition to physical materials. Across the studies, children could solve fraction problems by moving physical materials, even though they frequently could not solve the same problems in their head, even when shown a picture of the materials. The two processes of PDL were adapting and reinterpreting the environment. As children adapted the physical situation, they moved from an interpretation of individual pieces to an interpretation of partitioned structures as groups that could be individuated and counted in their own right. In addition, after initial learning, the ability to repurpose new environments to support cognition was a marker of strong understanding. Initial opportunities to adapt and reinterpret environments led toward this form of understanding, whereas opportunities to operate in environments that precluded adaptation and reinterpretation did not.

### *10.2. Effects of adaptation and reinterpretation*

Experiment 1 demonstrated that manipulation helped children solve fraction problems that they could not solve when looking at pictures of the same pieces. When the children could not manipulate the pieces, they tended to use a pencil to circle pieces that they interpreted as whole numbers. When the same children could manipulate the physical materials, they eventually made partitioned structures they could interpret as groups. The adaptation coevolved with the reinterpretation. Children did not simply implement a preconceived strategy with the pieces. They often started over within a problem, moving the pieces around until they could interpret the problem in a satisfying way.

Experiment 2 showed that the process of manipulating the pieces into a partitioned structure was important for reinterpretation. Preorganized pictures of the pieces did not help the children develop good interpretations as much as manipulating the pieces. For the younger children of Experiment 2, the benefit of manipulation depended on eliciting the prior knowledge of sharing fairly, which guided their activity. The data also showed that action per se is insufficient. In Experiment 2, children took more actions using pencil and paper than they did with the physical materials, but this increased activity did not propel them to greater insight.

Experiment 3 bracketed PDL. When children had relatively high knowledge (multiplication), they could use many types of materials to distribute their cognition. They were not developing new interpretations, but rather, they were repurposing environments to serve their interpretations. In contrast, when children had relatively low knowledge (fraction addition), the structure of the environment shaped their behaviors, and they were unable to develop new interpretations. The children only created appropriately partitioned structures with certain types of materials (e.g., pie pieces), and they were unable to interpret the resulting structures to solve the fraction problems.

Experiments 4 and 5 demonstrated that the benefit of PDL depends on the structure of the manipulable environment. The structure of the manipulatives had an effect on children's ability to transfer fraction addition to new situations with new physical properties. More important, it showed that environments with supportive problem-solving structures can lead to less learning than environments that require active adaptation and reinterpretation. The opportunity to adapt tiles and reinterpret them as groups and parts of wholes helped children handle new situations better than the children who worked with pies. The pie materials more readily offered an interpretation of groups and wholes, but as a consequence, the children did not learn how to make and interpret new grouping and whole structures, and they could not handle new situations with different physical characteristics.

In combination, the results suggested that simply seeing a final good structure is insufficient for children to develop new interpretations. Distributed activity was necessary to allow adaptation and interpretation to coevolve, so that the children could appreciate the mathematical meaning of a good physical structure once it appeared.

### *10.3. Toward a model of PDL*

PDL raises interesting methodological and explanatory challenges. In many experiments, investigators can begin with the assumption of a fixed stimulus that does not change during the process of learning, problem solving, or development. In this work, children changed the stimulus materials both physically and interpretively. The five studies showed that children changed the structures made by the materials and in the process changed their interpretations of those structures. For the developing child, the environment (or stimulus) is not a given; for the researcher, the structure of the situation becomes a relation between child and environment, such that changes to one change the other.

Often, it is theoretically safe to view the environment as fixed and to document changes to the individual in relation to that environment. However, when people can fundamentally adapt the environment and the interpretations that the environment supports, process accounts that only model a person's thoughts and that assume a consistently structured world become cumbersome. One alternative is to abstract the environment and person as components of a larger

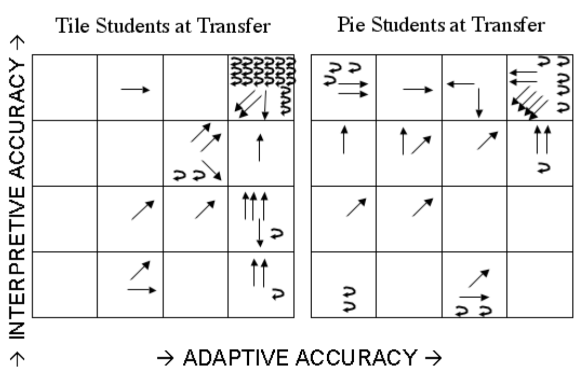


Fig. 12. Movement through the space of interpretive and adaptive accuracy in the transfer phase of Experiment 4. A straight arrow indicates a change in either adaptive or interpretative accuracy from one day to the next. A diagonal arrow indicates a change in both adaptive and interpretive accuracy. A curved arrow indicates no change in either adaptive or interpretive accuracy. See text for a detailed explanation of the figure.

dynamic space. Fig. 12 provides one approach using the transfer data from Experiment 4. This figure requires some explanation.

During the transfer phase each day, children completed problems at the levels they had attempted during the learning phase. For each problem level, they used three transfer materials. The horizontal axis in Fig. 12 represents the number of partitioned physical structures they made across the three transfer materials for a given problem level (from 0 to 3). The vertical axis represents the number of accurate interpretations (answers) for each of the three transfer materials. The 16 regions in the graph represent different ratios of adaptive to interpretive accuracy. To help read the figure, imagine it only showed a single point. The point would stand for a student's performance across the three materials for a single problem level on a single day. It would represent the student's success in adaptive restructuring relative to interpretive success. A point in a region on the diagonal would represent that the student correctly partitioned and interpreted the materials at the same rate; a point in the upper right corner would mean 100% correct on both. A point below the diagonal would indicate the child made more good adaptations than interpretations. A point above the diagonal would indicate the child made more good interpretations than adaptations. Fig. 12 uses arrows rather than points to indicate trajectories of change. The arrows represent performance changes between pairs of days. The tail of an arrow indicates the region a student was in for a given problem level on one day, and the head of the arrow points to the region the student attained on the next day for the same level of problem. Children typically moved to adjacent squares, with only 20% jumping across a square (omitted to simplify the figure). Circular arrows mean there was no change between days.

The sample is too small for conclusive statistical analyses of these results. The graph often has only 1 to 3 participants per region. However, the adaptive–interpretive space reveals some interesting patterns.

1. Begin with the plot for the tile children. One thing to notice is that most of the children adapted the environment better than they interpreted it before they reached full understanding; they are generally at or below the diagonal.
2. The second thing to notice is that the upper right region of 100% performance is a strong attractor state. The flow of arrows leads through the physical structure to the upper right corner. Once children enter the region, they rarely leave on subsequent trials.
3. Next, consider the plot for the pie children. In this case, the plot shows more movement toward and through the upper diagonal of the space. Children's interpretations are driving the performance of the system. Also, the upper right region is not a strong attractor state. It does not always draw the children from the remote regions; fewer children end in the region; even when they arrive, they are likely to leave.

Compared to the tiles, learning with the pies does not yield a stable and directed adaptive–interpretive space when children transfer to new materials. Across the two graphs, physical structuring leads the way toward stable and full understanding, whereas correct interpretations do not. Evidently, a strong pathway to knowledge comes through the ability to make adaptations that are just ahead of one's interpretations. As fits our PDL story, people manipulate the environment until a structure emerges that they can interpret meaningfully.

### *10.4. Development and educational possibilities*

These experiments were not instructional experiments. They examined how children solved problems, learned, and transferred in individual settings. This approach misses important factors, including social influences, classroom norms, and task structures that influence how manipulatives will impact learning. At the same time, studies such as these can illuminate particular ways that manipulatives influence children's action and thinking. This information should be able to inform instructional practice and vice versa.

The primary implication for learning that we draw from this research is not that children should only manipulate tiles to learn about fractions. It seems likely that working with many representations and materials is best for overall development of fraction understanding, and perhaps computer "virtual" environments that combine the abilities to move and draw on pieces could offer more value than real manipulatives (Ainsworth, Bibby, & Wood, 2002; Clements, 2002; Moyer, Bolyard, & Spikell, 2002). Instead, our account of PDL suggests a rationale for the use of manipulatives that is complementary but different from current views on their use. Researchers support several alternative views but have little agreement on why manipulatives might benefit learning (Chao et al., 2000). One idea is that exposure to multiple representations generates better understanding of underlying mathematical principles (Ainsworth et al., 2002; Moreno & Mayer, 1997). Another hypothesis is that a manipulative's structure should instantiate important mathematical concepts; for example, Cuisenaire rods demonstrate the base-10 system of numeration (Fuson & Briars, 1990). Another view is that external resources primarily help problem solvers keep track of problem elements without wasting internal memory resources (Cary & Carlson, 1999).

Our view of PDL is that the coupled processes of adapting and reinterpreting the environment drive quantitative development. So, rather than assuming that the quantitative meaning of a physical situation is manifest for children, the process of adapting the situation to develop an interpretation may best prepare children to understand quantity. This implies that simply showing children how to use a manipulative to solve a mathematical problem does not guarantee they will develop appropriate interpretations, and in fact, it may block the adaptation–reinterpretation activity. Instead, it may be better to provide children a chance to grapple with structures and interpretations that can prepare them to learn subsequently (Bransford et al., 1989; Schwartz & Bransford, 1998). In Experiment 4, for example, children inferred the appropriate steps for solving the problem over 50% of the time when they simply received feedback that showed the desired end state. Similarly, children in Experiments 4 and 5 were more prepared to learn how to use new materials when they actively grappled with the tile structure instead of working with the pie structure. The pie structure circumvented the need for adaptation and reinterpretation.

It may seem inefficient to have children waste time developing inappropriate structures and interpretations—why not just tell or show them the correct procedure to start with? One answer is that giving the solution before children have actively engaged in the quantitative challenge prevents the children from learning as deeply, for example, when measured by their ability to transfer (Hatano & Oura, 2003; Schwartz & Martin, 2004). The activity of adaptation and reinterpretation in distributed learning is an important element of development. It prepares children to appreciate the meaning of the solution when it appears.

The pedagogical challenge is to determine the appropriate level of scaffolding for the process of PDL. Based on this work, useful scaffolds should help a child find and work with critical aspects of a problem, without doing the work for the child. An analogy comes from training wheels on a bicycle. If the wheels constantly touch the ground, like a tricycle, the child may never learn to balance. However, by keeping the wheels slightly above the ground, the child has to struggle with the critical element of balance. For learning about quantity, good scaffolds need to support key adaptations and reinterpretations. For example, working with a whole block that could not be subdivided would not support creating the adaptations needed to solve a problem such as ⅓ of 12. However, if a scaffold carries too much of the burden, it interferes with the student's active process of struggling with the ideas. For example, if the material is subdivided into thirds, the student does not need to think about what 1/3 of 12 means.

Manipulatives can help children learn because they provide an environment for quantitative activity in which children can adapt and reinterpret. The assumption is not that children will necessarily discover the correct interpretation but, rather, that the activity will prepare them to understand the interpretation of a structure once it becomes available. This may be the hidden value of hands-on activities: They prepare children to learn from new resources, perhaps from a physical demonstration or a lecture, even in those cases when children do not generate the conventional solutions for themselves.

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