On People's Incorrect Either-Or Patterns in Negating Quantified Statements: A Study

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Abstract

People manifest formally incorrect either-or, polarizing response tendencies when asked to negate quantified statements. Our study focuses on students' error patterns when negating quantified sentences, which are the single most important cause for their difficulties with indirect proofs and proofs by contradiction. We found that, contrary to our expectations, the effect of content is relatively small on their negation behavior; that of the four quantifier categories used, students have by far the most difficulties in negating universally quantified sentences; and that the effect of formal logic instruction wears off relatively fast. The significance of our study reaches beyond the classroom: logically literate reasoners are less prone to be manipulated by either-or rhetoric of politicians, and are more conscious of their own limiting beliefs.

Keywords: negation of quantified sentences, polarizing tendency, content effect, effect of formal logic instruction, either-or rhetoric, limiting beliefs.

The Problem

"No" is one of the first words a child hears. And yet, negation is a logical concept that has puzzled philosophers and logicians for 2,500 years. As Horn (1989) notes, "negation is to the linguist and linguistic philosopher as fruit to Tantalus: waving seductively, alluring palpable, yet just out of reach, within the grasp only to escape once more" (p. xiv). Although a great body of work has emerged since the 1950s in cognitive science exploring people's difficulties in reasoning with negation as part of their deductive reasoning, little work has been done on *how* people actually negate given statements, especially quantified ones.

Theories of Deductive Reasoning

Inferences based on formal logic set the norm for people's deductive reasoning—they are viewed as the way people *ought* to reason. Many cognitive scientists have been attracted to the doctrine of *mental logic* (Braine & O'Brien, 1991; Rips, 1994), which claims that the human mind contains a reasoning system like that of formal logic. However, in a brilliant and by now classical experiment called *Wason's selection task* (Wason, 1968), Wason showed that this is not the case. Specifically, he showed people's difficulty reasoning with inferences in the form known as *modus tollens* (If A then B, not B $\therefore$ not A). Theories trying to explain/predict people's deductive reasoning behavior abound. Most of them agree that reasoning is rule-based but content-dependent, in particular it is dependent on content that evokes relevant knowledge from memory (Manktelow & Over, 1990).

People make systematic errors in deductive reasoning that are not predictable by any of the rule-based theories mentioned above. Many of these are successfully explained and predicted by the *mental model theory* (Johnson-Laird, 1983; Johnson-Laird and Byrne, 1991) that "postulates that reasoning depends on understanding the meaning of premises, and then using this meaning and general knowledge to construct mental models of the possibilities under description" (Johnson-Laird, 2001, p. 5). Recently, a large body of papers on mental model theory has explained and predicted errors ("illusions") that people make in reasoning by showing that they construct incomplete (or not fully explicit) mental models of the premises (e.g., Sloutsky & Johnson-Laird, 1999; Goldvarg & Johnson-Laird, 2000; Yang et al., 2000; Yang & Johnson-Laird, 2000a, 2000b; and Johnson-Laird, 2001). One of the most powerful explanations/predictions stems from the so-called *principle of truth*, which states that people have great difficulty mentally representing what is false according to the premises. This is particularly true in the case of quantified reasoning (Yang & Johnson-Laird, 2000a, 2000b).

People Misconceive Formal Negation

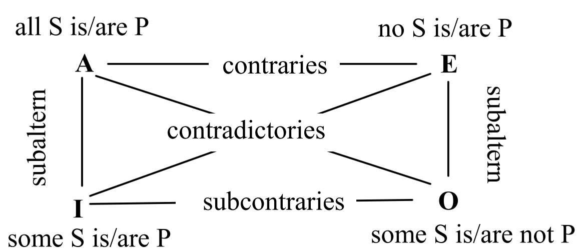
In this paper we show that in the case of quantified reasoning, not only do people not build mental models of negations, but more fundamentally, *they often don't know how to negate sentences correctly* in the first place! In particular, we show that they tend to mistake contrary opposition for negation (which is defined as contradictory opposition—see Figure 1 below), and their negation behavior is often automatic, including a "blind" polarized response.

People's Tendency Toward Polarized Thinking

Neither rule-based nor mental model theories of reasoning explain/predict people's negation behavior, particularly the so-called "*centrifugal politics and theology of polarization*" (Horn, 1989, p. 270) or either-or thinking.

Formal negation is defined as *contradictory opposition*. The A/O and I/E pairs in Figure 1 below are *contradictories* because in any situation one member of each must be true and the other false. The A/E pair is a *contrary* opposition because in some situations both can be false; their contradictories (O and I, respectively) can sometimes both be true of the same subject; e.g., “Not every rose is red” (or “Some roses are not red”) and “Some roses are red.” (Horn, 1989, pp. 10-11).

Contradictories such as “All mammals live forever”/“Some mammals do not live forever” or “No mammals breathe”/“Some mammals breathe” do not allow any middle possibility, while contraries such as “All kids are lazy”—“No kids are lazy” do, formally, allow middle possibilities: some kids may not be lazy and some kids may be lazy. In practice, however, naive reasoners tend to void the space between contraries, *turning them into either-or disjunctions* (“Either all kids are lazy or no kids are lazy”). A similar phenomenon is observable with predicates. While contradictories such as black/not black, odd/even exclude any *middle term*, contraries, such as black-white, short-long, empty-full, in principle do not; your pants may be neither black nor white. “But the context may fill in the gap between the contraries, establishing a disjunction of the type normally associated with contradictories. The middle is not so much excluded as pragmatically absorbed, and p or q becomes and instance of p or not-p. One context triggering this absorption is the black-or-white, centrifugal politics of polarization (‘He that is not with me is against me,’ ‘If you’re not part of the solution, you’re part of the problem’)” (Horn, 1989, p. xviii).



A: all monkeys are slow
I: some monkeys are slow
E: no monkey is slow
O: not every monkey is slow, some monkeys are not slow

Figure 1: Traditional Square of Oppositions (Horn, 1989, p. 208)

The naive reasoner who is not trained in formal negation, falls easily pray to this polarizing tendency, to “this drift of middle-allowing lexical contraries into middle-excluding acting contradictories” (Horn, 1989, p. 271). Sapir (1944, p.133, cited in Horn, 1989, p. 271) explains: “A speculative mind might attribute the polarizing tendency to the presumed survival value for the primitive language wielder in perceiving and classifying the universe into various series of either-or, black-or-white, or ... yin-or-yang pairs, ignoring the quasi-scientific niceties of the zone of indifference.”

A similar tendency is manifested with sentences of the form, “Some S are P” such as “Some horses are slow,” which is often negated as “Some horses are not slow,” or even as “Some horses are fast” (polarizing tendency). People often turn the disjunction “Some S are P or some S are not P” into an either-or disjunction, mistaking “Some S are not P” for the contradictory opposition (i.e., negation) of “Some S are P.”

Why Study Negation of Quantified Statements?

Psychology of Either-Or Language of Over-Generalized Beliefs “I never seem to do the right thing,” “Everybody hates me,” “Nobody ever tells me anything,” or “Math [all of it!] stinks” are some of the familiar sentences that express unqualified or over-generalized beliefs (Hale-Haniff & Pasztor, 2000). They presuppose that there are *no exceptions*, and therefore, *no choice*; the implication is that the statements must always be true.

Over-generalized beliefs are most often uttered by people who feel stuck or depressed. When we are stuck and/or depressed, when flow temporarily stops in our experience, we seem to forget that the negation of “I never enjoy anything” is “I sometimes enjoy something” [be that even just a candy or a hot shower], as opposed to “I always enjoy everything.” It is often the task of successful therapy to help people recognize whether their belief statements reflect useful or necessary rules or generalizations that need to be updated or applied to a narrower context than everywhere and all the time.

Mathematics Education Over decades of teaching mathematical logic, the first author has observed that students experience serious difficulties formally negating quantified statements and therefore doing indirect proofs and proofs by contradiction. Formal logic instruction doesn’t seem to improve significantly students’ ability to negate quantified statements or do indirect proofs or proofs by contradiction. This phenomenon has also been documented by others, such as Barnard (1995, 1996), Thompson (1996), and Antonini (2001), although research on negation in education has been sparse.

Barnard (1995) identified factors that influence students’ negation behavior, such as logical structure, lexical representation, context, level of abstraction, and degree of complexity. He found that even after two years of studying, about one third of the students had difficulty in negating even simple statements. He also found that students with less training in abstract reasoning are more likely to be influenced by the truth value of the statements, especially if these statements are set in everyday contexts. In a follow-up study, Barnard (1996) reinforced his earlier findings and also found a strong tendency, especially among younger students, “to avoid ranges of variation (such as considering ‘some’ as being between ‘all’ and ‘none’) and subdivisions” (p. 142).

Thompson (1996) studied the difficulties that students had with indirect proofs. Even after instruction, she found that only less than one third of the students were successful in negating conditionals and thus performing indirect proofs.

Antonini's (2001) study provides insights into why proof by contradiction is difficult for many students, pointing to the differences between natural language and mathematical negation.

These studies document that mathematics education is in dire need of ideas and tools that can help design more effective instructional procedures to teach negation. As most theorems are stated as quantified statements, understanding factors that improve students' negation behavior of quantified statements can also improve students' performance in proofs by contradiction and indirect proofs in mathematics.

Provisional Summary The psycholinguistics study of people's negation behavior is of great importance in mathematics education, but also reaches into the psychological, philosophical, and political bases of our every-day lives. The logically literate citizen is not only able to successfully prove mathematical statements, but is also less prone to be manipulated by the either-or rhetoric of politicians, and is more conscious of his/her limiting beliefs.

The Study

Guiding Questions

Three basic questions guided our study: **1.** How do students' performances compare in negating statements with different quantifiers and statements with different semantic structures? **2.** How do the performances change immediately after instruction of formal negation? How robust are such changes, if there are any, i.e., to what degree do they last after a prolonged period of time after instruction? **3.** What are the incorrect schemes of negation that students use, and how do these schemes change immediately after formal instruction? How robust are such changes?

Methodology

Instrument For this study, we developed an instrument with 16 sentences. The sentences consisted of statements containing four types of quantifiers: *all, none*, *some*, and *some ... not*. Further, each quantifier was cross-matched with the following four types of semantic categories: **A.** abstract sentences, **B.** nonsensical sentences, **C.** meaningful but false sentences with true negations (designed to serve as control for effect of content), and **D.** meaningful sentences, the negations of which are false (such as in the case of "Some animals don't live under water," the negation of which is "All animals live under water") or, if they are true (as in the case of "All mammals live forever," the negation of which is "Some mammals don't live forever"), they provide *weaker* true information than the contraries (in this case, "No mammals live forever") (c.f. Horn, 1989, p. 211).

The sixteen statements were placed in a list in random order to avoid order effect. Figure 2 gives the statements included in the instrument. In the first row, we included in parentheses the correct formal negations (for ease of understanding of the data analysis).

Participants Thirty two undergraduate college students taking "Logic for Computer Science" and fifteen computer science Master's and Ph..D. students in a "Cognitive Science" class participated in the study. The graduate students were chosen from among those who had taken an undergraduate logic class at least one year before the study.

Procedures Participants were administered the instrument at the beginning of the fall term before the formal instruction of negation (pretest), and at the end of the term before final exams (posttest). The purpose was to study *how* students negate quantified statements before formal instruction and to see how this changes after such instruction. Graduate students (graduate group test) provided a measure of negation behavior after a longer time has elapsed from the time of formal instruction.

A. Abstract statements	1. All x's are y. (Some x's are not y.) 2. No p is q. (Some p's are q.) 3. Some i's are j. (No i is j.) 4. Some r's are not s. (All r's are s.)
B. Nonsensical statements	1. All morgies are brig. 2. No borogove is mimsy. 3. Some tambas are pasty. 4. Some cirks are not trique.
C. Meaningful statements with true negations	1. All teenagers are lazy. 2. No book is boring. 3. Some machines are alive. 4. Some living things do not grow.
D. Meaningful statements with negations that are false or provide weaker true information than contraries	1. All mammals live forever. 2. No mammals breathe. 3. Some animals live on earth. 4. Some people do not live underwater

Figure 2**:** Categories of the statements contained in the negation instrument.

Participants were asked to work on the instrument during class time and individually at their own pace. They were given class participation points. They were asked to write the "negation" of each sentence, where the *negation* of a sentence A was defined as a *contradictory* sentence B that denies the truth of the given sentence A, and in every situation, one of A and B must be true and the other false.

Data Analysis Data was analyzed both quantitatively and qualitatively. First, students' negation of each sentence was evaluated for correctness for each administration of the instrument. Results were tabulated by the type of quantifiers and by the semantic structure. This allowed us to see the level of difficulty of statements with different quantifiers and semantic structures. To ensure scoring reliability, two independent coders each scored the same subset of student responses, and an acceptable level of reliability was observed. Further, common schemes of errors made in the negations were analyzed qualitatively, and findings were again organized by quantifier type and semantic structures.

Results from Quantitative Analysis of Students' Responses

Table 1 presents students' success rates in negating statements with different quantifiers and different semantic content. The rows present the rates for statements with different quantifiers in the undergraduate pretest and posttest, and the graduate group test, respectively. The columns give the rates for the statements in the four semantic categories.

The *pretest* results reveal that overall, less than one third of the students (28%) negated statements correctly. For quantifier categories, the lowest success rate was for "no" sentences at 16%, and the highest was for "some" sentences at 42%, resulting in a *range* of 26% (highest minus lowest rate). Pretest success rates in the semantic categories, however, were quite uniform, with a small range of only 3%.

Table 1: Percentages of correct responses by categories of statements (1: n=32; 2: n=15)

Categories		A	B	C	D	Avg.
All	*pre*[1]	32	21	25	25	26
	post[1]	66	69	72	63	68
	grad[2]	33	40	40	40	38
None	*pre*	14	18	11	21	16
	post	75	81	72	78	77
	grad	33	33	33	33	33
Some	*pre*	39	46	43	39	42
	post	66	78	75	69	72
	grad	67	73	73	67	70
Some ... are not	*pre*	25	36	29	25	29
	post	66	72	66	63	67
	grad	60	53	67	60	60
Average	pre	28	30	27	28	28
	post	68	75	71	68	71
	grad	48	50	53	50	50

Not surprisingly, in the *posttest*, i.e., after students received formal instruction on negation, the overall rate of success more than doubled (71%). The range of success in quantifier categories fell to about 10%, and the highest rate of improvement was in the "no" statements. For semantic categories, in the posttest the range stayed at the low level of 7%.

The overall *graduate student* performance rate was about 50%. The range was once again much higher in quantifier categories (37%) than in semantic categories (5%). As in the pretest, the "no" statements were the most difficult to negate for this group (33%), which were followed by the "all" statements (38%).

Regarding semantic content, we expected the "C" type sentences to be the easiest to negate correctly. This was not the case. We also expected the "D" type sentences to have the lowest rate of correctness, which, again, was not the case. The semantic categories did not seem to be associated with differential difficulty levels. The range was consistently small in all three measurements.

Although instruction did seem to make an immediate difference, in the graduate group test the effects of instruction faded considerably for the statements with "no" and "all" quantifiers.

To summarize, results from quantitative analysis seemed to support our expectation that students would have more difficulties in negating universally quantified sentences (all and none) than other types. Results also supported our expectation that the effect of formal instruction, which is predominantly *verbal* (c.f. Pasztor, 2003), is not robust, fading away significantly after a year. However, data did not support our expectation that semantic content of sentences plays a significant role in students' negation behavior.

Results from Qualitative Analysis of Student Responses

For this analysis, we identified the students' *schemes of negation*, including their patterns of incorrect negation, and found their rates of use in the pre-, post-, and the graduate group test, respectively. This analysis provided a measure of the effect of formal instruction on the ways students thought while negating the statements. It also gave a measure of which schemes were robust by virtue of the rate at which we observed graduate students using them even after a year or more had elapsed since formal instruction.

Table 2 presents the eleven schemes that we found and the rates at which students applied them in each test, respectively.

Table 2: Rates of the schemes of negation

categories	pretest % (n=32)	posttest % (n=32)	graduate class % (n=15)
contrary: all → no	71	16	60
contrary: no → all	71	13	67
subcontrary: some → some ... not	54	22	33
subcontrary: some ... not → some	46	22	33
mechanical response	36	75	40
dropping quantifiers	36	16	7
subaltern: e.g., some ... not → no	14	22	0
trivial negation	14	13	7
attaching personal meaning to words	14	3	7
truth value effect	11	16	7
reformulation	11	0	0

Some of the negation schemes require explanation:

- Mechanical response: Various response categories showed that students were mechanically applying syntactic rules/patterns.
  - Use of "there is [or exists] at least one..." For example, "There exists at least one book that is boring" instead of, for example, "Some books are boring."
  - Use of "All ...are not ..." For example, some students negated "Some machines are alive" as "All machines are not alive," instead of, say, "No machines are alive." They seemed to be automatically applying the formula "Some ..." → "All .... not ...." These students usually did not use the quantifier "none" at all.
  - Use of "No .... don't ...." For example, some students negated "Some living things don't grow" as "No living things don't grow," instead of, say, the good old English sentence, "All living things grow." They seemed to be automatically applying the formula "Some ..." → "No ...."
- Dropping quantifiers: A number of students negated "No ..." type sentences by simply dropping the "no." E.g., they negated "No p is q" as "p is q" and/or "No books are boring" as "Books are boring."
- Trivial negation: Some students used "prefix" negation, i.e., they negated a sentence by putting "not" in front of it. Interestingly, they did this only with "All ..." sentences, which shows that after all, it did matter to them whether negations sound/look natural.
- Attaching personal meaning to words: Some students found personal (if sometimes "incorrect") meaning in the words, i.e., they didn't treat them as mere symbols. For example, they wrote "dead' instead of "not alive," "die" instead of "not live forever," and "on land" instead of "not under water." One student commented the sentence, "Some machines are alive," with "Really? AI?"; another student correctly negated the 4D sentence, "Some people don't live under water," by "All people live under water," but then humorously commented on the semantic content by "merpeople?"; and one student wrote "hardly any" for some instances of "no," and "mostly" for some instances of "all."
- Truth value effect. There were two kinds of evidence for the effect of the truth value of the negation of the statements.
  - Category C sentences were correctly negated, whereas other category sentences of the same structure were not. For example, a student negated the 4C sentence, "Some living things don't grow," as "All living things grow," whereas all her previous "Some ...." sentences were negated incorrectly. Another student negated the 3C sentence, "Some machines are alive," as "No machines are alive," even though all her other "Some ...." type sentences were negated incorrectly; moreover, this was the only time the student used the quantifier "no."
  - Category D sentences were incorrectly negated, while other category sentences of the same structure were negated correctly. For example, some students who negated all other "All ..." sentences correctly, negated the 1D sentence, "All mammals live forever," incorrectly as "No mammals live forever;" and the 2D sentence, "No mammals breathe," as "All mammals breathe," although all other "No ..." type sentences were negated correctly. Some students had all negations correct, except the negation of the 4D sentence, "Some people don't live under water," which they negated as "No people live under water," instead of "All people live under water." In fact, most students who were influenced by the truth value of the negation, were misled by this sentence.
- Reformulation: Some students reformulated the sentences instead of negating them. For instance, they negated "All students are lazy" as "No students are not lazy."

For the pretest, the most frequent misconception was that negations are contraries. Specifically, "All" and "No" sentences had a 70% rate of contrary response. This was followed by a rate of about 50% for subcontrary responses for "Some" and "Some … not" sentences. Although the rates of contrary and subcontrary responses decreased significantly immediately after instruction, for the graduate class they climbed back up to be the highest (not counting "mechanical response"). However, it is worth noting that the rate of subcontrary responses was half of the rate of contrary responses for the graduate class. Consequently, it is possible to conclude that the tendency to negate by contraries is much more robust than the tendency to negate by subcontraries.

Mechanical responses were observed in about one third of students in the pretest, and it more than doubled in the posttest.

The effect of semantic content (truth value effect, and effect of the meaning of words) stayed at a relatively low rate in all three measurements. The truth value effect was 1 in 10 in the pretest, 1 in 7 in the posttest, and almost negligible in the graduate group test. This parallels our observation in the quantitative analysis of data, that the range of rates in semantic categories was quite small. Our hypothesis that semantic content of sentences affects students' responses was therefore not supported.

Discussion and Conclusion

Neither rule-based theories nor mental model theory have been concerned with *how* people actually negate sentences, particularly quantified ones. In the present paper, we have focused on students' negation behavior of quantified statements before and after formal instruction of negation.

The results of our study have corroborated the so-called "centrifugal politics and theology of polarization" (Horn, 1989, p. 270) and tendency for automatic negation responses. Specifically, our qualitative analysis of responses showed that most commonly, students conceive of contrary and subcontrary opposition (see Figure 1) as negation of quantified sentences. The contrary opposition as negation scheme was observed in more than two thirds of responses in the pretest. Although it fell dramatically in the posttest, in the graduate group test it rose back up to about two thirds of all responses. Negating quantified sentences by applying mechanically or "blindly" syntactic rules was the most common type of response behavior observed after the use of contrary and subcontrary opposition schemes in the pretest. It had the single highest rate in the posttest. Although our study did not look into the nature of instruction students received between the pretest and the posttest, this high rate suggests that instruction of formal negation may actually foster a mechanical approach to negation.

We were able to observe some personal meaning making as evidenced in the "truth value effect" and "attempts to attach personal meaning to words" of our qualitative data analysis, although at a very small rate. We suspect that this low rate can be attributed to the format of our instrument and the context in which it was administered. Our instrument consisted of a small number (sixteen) of unrelated statements, given in a list format. This might have encouraged students to look for syntactic rules for negating structurally similar sentences. Furthermore, the instrument was administered in a classroom setting, into which students usually come leaving their personal sense-making behind, expecting to acquire and be assessed for procedural as opposed to conceptual knowledge (Alacaci & Pasztor, 2002).

In a more recent study (Alacaci & Pasztor, 2005), in which we used an instrument that describes a coherent scenario providing a semantic link between the sentences, and that was administered in a semantically rich context, we were able to observe significant effects of content on students' negation behavior.

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