A Microgenetic Study of the Conceptual Development of Inversion on Multiplication/Division Inversion Problems

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Abstract

The purpose of this study was to conduct a microgenetic study of the development of the concept of inversion as it applies to multiplication and division inversion problems. The study was modelled on Siegler and Stern's (1998) study in which Grade 2 participants solved addition and subtraction inversion problems (*a* + *b* - *b)* for 6 weekly sessions. In session 7, modified inversion problems (*b* + *a* - *b*) as well as lure problems (*b* - *a* + *b*) were also included. In the current study, Grade 6 participants solved multiplication and division inversion problems (*d* x *e* ÷ *e)* during 6 weekly sessions. Previous research has shown that this latter type of inversion problems is more difficult than the former type. The present results indicate that there are differences in how frequently participants discover and apply the inversion concept compared to Siegler and Stern's (1998) work. The findings add to the recent body of knowledge indicating that the concept of inversion as it applies to multiplication and division is significantly more difficult than it is for addition and subtraction.

Keywords: arithmetic; inversion; conceptual knowledge; microgenetic.

Introduction

In the domain of mathematical cognition, it has been historically difficult to assess conceptual knowledge, and in particular the development of conceptual knowledge (Bisanz & LeFevre, 1990). Conceptual knowledge is the understanding of the underlying structures of mathematics (Bisanz & LeFevre, 1990). Recently, there has been an increasing interest in the development of conceptual knowledge (e.g., Rittle-Johnson, Siegler, & Alibali, 2001) and yet it is often difficult to directly assess children's understanding of the underlying structures of mathematics. One type of task that has been successfully used in the past, however, is the *inversion problem* (Starkey & Gelman, 1982). These problems are of the form *a* + *b* - *b* and can be used to assess whether participants understand the inversion concept. Because addition and subtraction are inverse operations, no calculations are required to solve an inversion problem as the answer to the problem is simply the first number, *a*. This solution approach is called the *inversion-based shortcut*.

There are a number of advantages to using the inversion problem to assess conceptual knowledge or understanding. First, for participants who do not yet know a written numerical system (e.g., preschoolers), inversion problems can be demonstrated using manipulatives (Klein & Bisanz, 2000). Second, unlike many tasks that assess conceptual understanding, participants do not have to have the verbal abilities to explain their understanding but can instead demonstrate their understanding through problem solving. Finally, supporting evidence that participants are indeed using the inversion concept, as demonstrated via stating that the answer is the first number, can be obtained by using *standard problems*. If a participant is simply stating that the answer is the first number, *a*, then the answer will be incorrect on standard problems of the form *a* + *b* - *c*.

Using inversion and standard problems, researchers have found that children, even preschoolers, can make use of the inversion concept to solve the inversion problems without any calculation and that inversion shortcut use increases across age (Bisanz & LeFevre, 1990; Bryant, Christie, & Rendu, 1999; Rasmussen, Ho, & Bisanz, 2003; Stern, 1992; Vilette, 2002). Siegler and Stern (1998) further extended the research on inversion shortcut use by examining the implicit and explicit components of conceptual understanding. They used a microgenetic design and found that by the end of the study Grade 2 participants were using the inversion shortcut over 90% of the time to solve inversion problems and all of the participants discovered the shortcut during the course of the study.

In more recent research, a new type of inversion problem has been investigated (Robinson & Ninowski, 2003; Robinson, Ninowski, & Gray, in press). This type of inversion problem makes use of the inverse relationship between multiplication and division and takes the form of *d* x *e* ÷ *e*. The same inversion-based shortcut can be used. Robinson and Ninowski (2003) compared adult performance on both types of inversion problems: Addition/Subtraction inversion problems and Multiplication/Division inversion problems. Adults used the inversion shortcut on both types of inversion problems but more frequently on the Addition/Subtraction problems (94% vs 85% for Addition/Subtraction and Multiplication/Division problems, respectively). In a following study, performance of Grade 6 and 8 students on both types of inversion problems was examined (Robinson, et al., in press). Once again, inversion shortcut use was much higher on Addition/Subtraction problems than on Multiplication/Division problems (44% vs. 19% in Grade 6 and 60% and 39% in Grade 8). Overall, the concept that multiplication and division are inverse operations appears to be more difficult for both adults and children than the concept that addition and subtraction are inverse operations.

The next step in the investigation of the concept of inversion as it applies to the operations of multiplication and division is to study in more detail how the concept is discovered and applied and to determine how discovery and application of the inversion shortcut compares to previous work on addition and subtraction inversion problems.

Present Study

In the current study, the development of the Multiplication/Division inversion concept was further investigated by using a microgenetic method. The same number of sessions and problem formats was used as in the Siegler and Stern (1998) study. The microgenetic design permits a more in-depth examination of the changes in performance across a relatively short period of time (Siegler & Crowley, 1991). In the present study, the focus was on the strategies that children used as measured by verbal report data. Verbal report data have been successfully used in previous research on arithmetic and, in particular, in inversion studies, and have been shown to be veridical (Robinson, 2001, Robinson & Ninowski, 2003, Robinson et al., in press). Verbal report data are very useful in a microgenetic design as children's discoveries, and in particular the verbalization of their discoveries, can be examined (Siegler & Crowley, 1991) .

Method

Participants

Forty-one Grade 6 students (mean age = 11 years, 6 months) were included in this study. Participants were drawn from a large Canadian city and were predominantly White and middle-class. To participate in the study, students were given a pretest in which they were presented with Multiplication/Division inversion (e.g., 9 x 6 ÷ 6) and standard (e.g., 9 x 8 ÷ 4) problems. If they did not report using the inversion-based shortcut to solve the inversion problems during the pretest, they participated in the study. Participants were randomly divided into two groups: blocked (n = 21) and mixed (n = 20). The study took place in the first half of the school year.

Materials

Both groups of participants completed seven sessions after the pretest. In the first six sessions, participants were asked to solve either 16 Multiplication/Division inversion problems only or 8 inversion and 8 standard problems. Participants in the mixed group solved both inversion and standard problems in all sessions. The blocked group received inversion problems only in Sessions 1, 2, 3, and 5 and both inversion and standard problems in Sessions 4 and 6.

In the final session, session 7, participants were asked to solve three types of three-term problems. First, participants solved four "original" inversion problems of the same format that they had seen in the previous sessions. Second, eight "modified" inversion problems were presented for which the inversion strategy would still be applicable (e.g., 8 x 6 ÷ 8, 21 x 7 ÷ 7) to see if generalization would occur. Third, twelve "lure" problems were solved to determine whether overgeneralization would occur (e.g., 4 x 2 x 2, 32 ÷ 4 ÷ 4, 30 ÷ 6 x 30). Thus, the inversion shortcut strategy was applicable on half of the 24 problems.

Procedure

Participants were individually tested once per week for eight consecutive weeks (from pretest to Session 7). Participants were presented with one problem at a time and asked to state their answer and how they had achieved that answer. Accuracy, solution latencies, and verbal report data was collected. All sessions were videotaped.

Results

The results reported here are based on performance on the inversion problems only and all significant differences are at least at an alpha level of .05.

Overall, reported inversion shortcut use increased from sessions 1 through 6, $F(5, 195) = 12.42$, $MSE = 509.06$. Though the trend was for the blocked group to have higher means for inversion use than the mixed group, the difference was not significant (see Figure 1). Both groups showed a steady and gradual increase but reported shortcut use remained well under 50% in both groups.

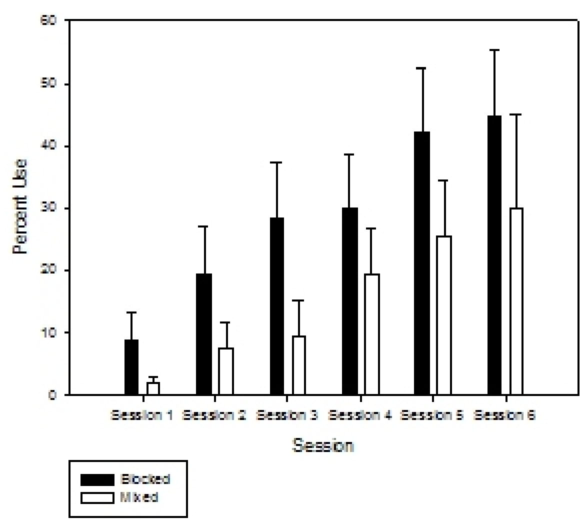

Figure 1: Inversion use across session for both conditions.

Accuracy and solution latencies improved across session, $F(1, 39) = 36.52$, $MSE = 74.93$ and $F(1, 39) = 27.68$, $MSE = 9.22$, respectively. The percentage of accurate responses grew from 91.3% to 99.4% over the course of the sessions while solution latencies decreased from 5.5s to 3.0s. If participants are using the inversion shortcut, there should be fewer errors and shorter solution times as the inversion shortcut, compared to a computational approach, is fast and usually error free (Robinson & Ninowski, 2003; Robinson et al., in press). The increased accuracy and shorter solution latency pattern found in this study is consistent with an increased use of the inversion shortcut but could also simply reflect an increased proficiency in the task. As with the percentage of reported inversion shortcut use, no group differences were found, although once again the trends were for performance to be better for the blocked group. Thus, the pattern is consistent with the tendency for inversion shortcut use to be higher in the blocked group.

Further analysis of the accuracy and solution latencies was conducted to provide corroborating evidence that the participants' verbal reports were veridical. If participants are using the inversion shortcut rather than a computational approach (e.g., 4 x 9 ÷ 9: calculate 4 x 9 then divide the product, 36, by 9), fewer errors should occur and solution latencies should be shorter. In the first and sixth sessions, for all inversion problems that both the mixed and blocked groups were given (n=8), the accuracy and solution latencies were calculated for inversion shortcut use and the use of computation. Analyses compared each session separately as different problems were used in each session. In both sessions, no differences in accuracy were found which is not surprising given the overall high accuracy and that no errors were made on the inversion shortcut trials. The trends were in the expected direction, however with the means for inversion shortcut higher than those for computation (see top of Figure 2). The expected pattern was found with the solution latencies, $t(8) = -3.84$, $SE = .69$ and $t(8) = -7.86$, $SE = .37$, for sessions 1 and 6, respectively. Inversion shortcut use was significantly faster than computation in both sessions (see bottom of Figure 2). These analyses provide evidence that participants were indeed using the problem solving solutions that they reported.

As a small number of students may have accounted for most of the reported inversion use, the percentage of students who reported using the shortcut at least once in a session is provided. As can be seen in Figure 3, the trend is for the number of participants using the shortcut to only gradually increase and then remain relatively stable across the final three sessions.

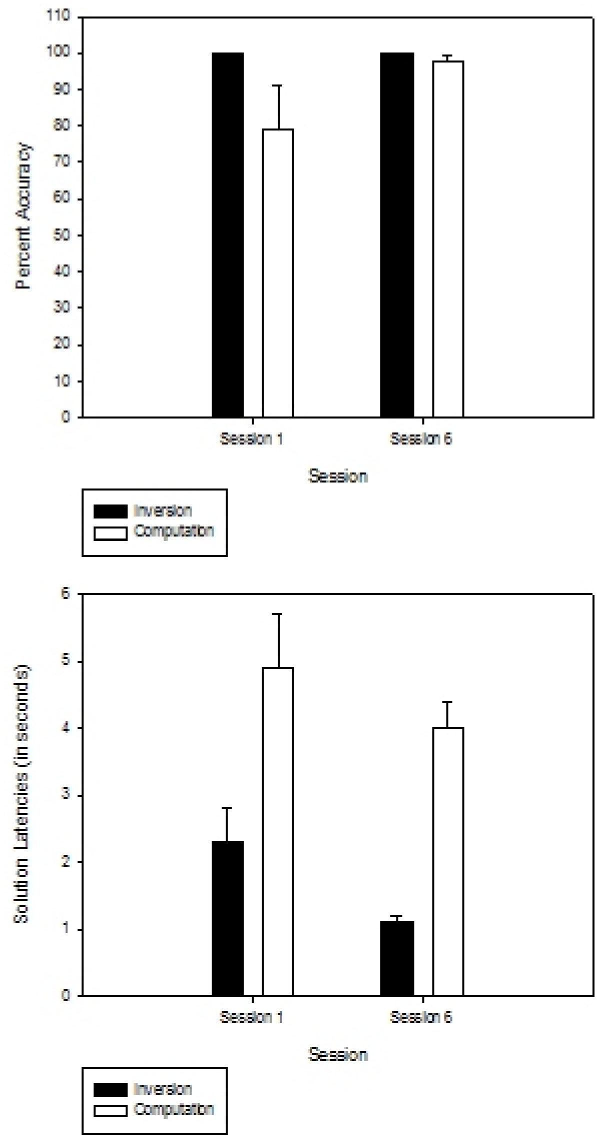

Figure 2. Accuracy and solution latencies for inversion use and computation.

Finally, as can be seen in Figure 4, the trend was for the blocked group to discover the inversion shortcut earlier than the mixed group. Based on Siegler and Stern's (1998) study, it was expected that by the end of the study at least 90% of the participants would have discovered the inversion shortcut. However, a significantly higher number of participants than predicted did not discover the inversion shortcut by the end of the sixth session, $\chi^2(1, 41) = 9.85$. No differences were found between groups but the trend was for the blocked group (71%) to be more likely to discover the inversion shortcut than the mixed group (60%).

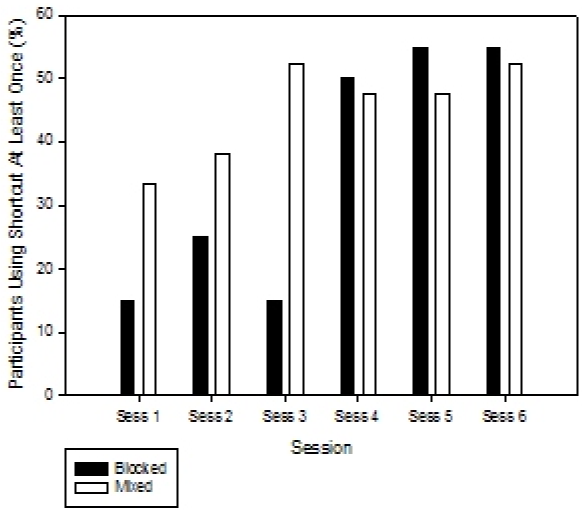

Figure 3: Students using shortcut at least once per session.

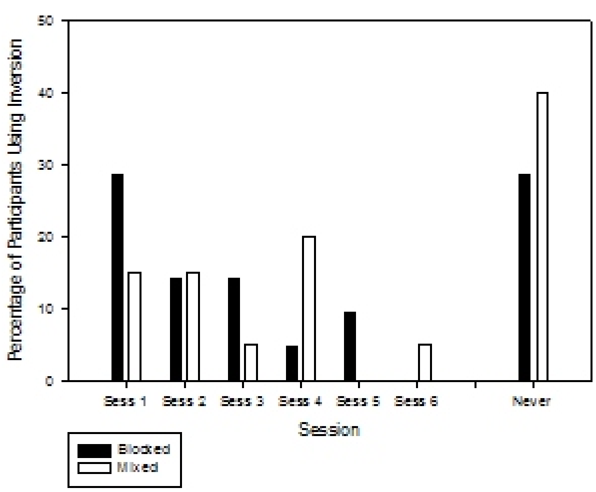

Figure 4: Session in which inversion was first used.

Verbal reports of the trial on which participants first reported using the inversion shortcut fit with Siegler and Crowley's (1991) assertion that strategy discoveries include a wide range of insight, awareness, and affect. In Table 1 a representative sample of participants' verbal reports on the trial in which they first reported using the inversion shortcut is included.

Table 1: Verbal reports of inversion shortcut discovery.

Example 1: Pp 616, Session 5, 5 x 4 ÷ 4

Experimenter (E): How did you get 5 so quickly?
Participant (P): (casual tone) aw... I just reversed them... got the same one... thing... and not even do the question (laughs).

Example 2: Pp 623, Session 3, 6 x 9 ÷ 9

E: How did you get that answer?
P: Um... I just knew it (sounds surprised).
E: Just right away?
P: (nods).
E: Is there anything that tipped you off that you just knew that it was 6?
P: Um... It has to be 6 'cause multiplication is the opposite of division.

Example 3: Pp 681, Session 4, 9 x 12 ÷ 12

E: How did you get the answer?
P: Um...
E: Did you know 9 x 12 or...?
P: No, I just reversed it.
E: Okay, so you just kind of...
P: Well, I didn't really know what 9 x 12 was so I just kind of thought 9.

Example 4: Pp 686, Session 4, 9 x 12 ÷ 12

E: How did you get 9?
P: Um... I looked at the first part and thought it would be backwards from the second part.
E: Oh, okay.
P: I was trying to figure out 9 x 12 and then thought "oh, that's too hard!"

In the final session, participants were asked to solve original and modified inversion problems as well as lure problems. On the original problems, inversion shortcut use was expected to be comparable to that of the previous sessions. On the modified problems, generalization of the inversion shortcut should lead to reported inversion shortcut use. On the lure problems, overgeneralization of the inversion shortcut would lead to reported shortcut use. As can be seen in Figure 5, there was a significant difference in problem format, $F(2, 78) = 13.30$, $MSE = 281.68$, with reported inversion use highest on the original inversion problems compared to inversion use on modified and lure problems that did not differ from each other. Although the difference between the blocked and mixed group was not significant, the trend was for original inversion use, generalized inversion use on the modified inversion problems, and overgeneralization of the inversion shortcut on the lure problems to be greater for the blocked group.

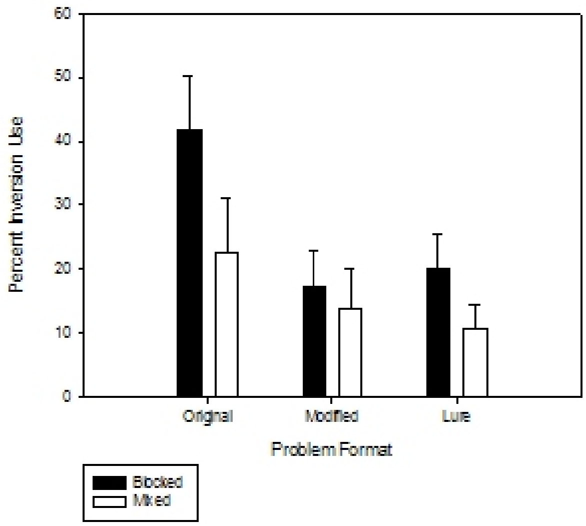

Figure 5: Inversion use on original, modified, and lure problems.

Discussion

Grade 6 students' reported use of the inversion-based shortcut increased during this microgenetic study, with the trend for more frequent shortcut use in the blocked condition. The expectation was that inversion use would be higher for the blocked group as they were exposed to only inversion problems rather than inversion and standard problems for four of the six sessions. This should have made the inversion concept and its associated shortcut clearer to the blocked group. Though the means were in the expected direction, this difference was not found as it was by Siegler and Stern (1998).

The inversion shortcut also never became the predominant problem solving strategy of choice as average reported use across all sessions was under 40%. In addition, many of the participants never discovered the inversion shortcut (34% total). These results are in sharp contrast to those of Siegler and Stern (1998) who found that shortcut use increased dramatically across their study. By the end of their microgenetic study, inversion shortcut use by Grade 2 students on Addition/Subtraction inversion problems exceeded 90%. Thus, in comparing the results from this study and that of Siegler and Stern it is clear that the concept of inversion is more difficult as it applies to multiplication and division than as it applies to addition and subtraction.

Further examination of the data in the first six sessions is needed to determine whether there is a marked difference in implicit (as demonstrated by faster solution times) and explicit (stating that the shortcut was used) understanding of understanding as was found by Siegler and Stern (1998) on inversion problems involving the operations of addition and subtraction. In this study, by examining the accuracy and solution latencies for the inversion shortcut compared to the computational approach, it was clear that, overall, participants were using the problem solving solution that they verbally report. Nevertheless, a more detailed examination may show the same pattern found by Siegler and Stern. In their study, reaction times in the inversion trials preceding the discovery of the inversion shortcut strategy seemed to indicate that often students had discovered the shortcut (as measured by their reaction times) before they were able to verbalize it. To offset this conclusion, however, it was clear from a number of the participants' verbal reports (e.g., Example 2 from Table 1) that participants were clearly surprised at their sudden discovery and immediately verbalized their newfound understanding of the inversion concept.

Finally, in the last session students had the opportunity to generalize and overgeneralize the inversion shortcut. Siegler and Stern (1998) posited that ideal performance would have been to use the inversion shortcut when appropriate and avoid it when inappropriate. If the inversion concept is well understood then the shortcut should be properly applied. However, Siegler and Stern found that correct generalization occurred infrequently in both the blocked and the mixed groups. More specifically, they found that although the blocked group used the inversion shortcut more often on inversion problems, the blocked group also used the inversion shortcut more often when inappropriate to do so. Thus, the blocked group transferred the concept or shortcut "wholesale."

In the present study, parallel results were found. Correct transfer or generalization was infrequent (15.5% across both groups) suggesting further evidence that the inversion concept as it applies to the inverse relationship between multiplication and division is poorly understood. No group differences were found in overall inversion use but the trend was for inversion use to be higher on both the original and modified inversion problems as well as the lure problems for the blocked group, not just on the original and modified inversion problems. Thus, the blocked group seemed to generalize and overgeneralize the inversion concept as a whole whether it was appropriate to do so or not. Taken with Siegler and Stern's (1998) results, this indicates that complete understanding of the inversion concept on both Addition/Subtraction and Multiplication/Division inversion problems may take further time to develop to allow for appropriate generalization or transfer. As Geary (1994) points out, if a concept is well understood then appropriate generalization should occur.

The current results, taken in conjunction with previous research on Multiplication/Division inversion problems, highlight the general difficulty that participants have with the concept of multiplication and division as inverse operations. Up until very recently, researchers investigating the concept of inversion with addition and subtraction inversion problems have concluded that the concept of inversion is one that is grasped even before exposure to formal arithmetic (e.g., Klein & Bisanz, 2000; Rasmussen et al., 2003). What is now becoming clear is that when considering the concept of inversion as it applies to the operations of multiplication and division, further research needs to be performed to examine why the same concept on a different pair of operations is less developed.

There may be a number of reasons why this inversion concept, as applied to multiplication and division, is so much more difficult for both adults and children. First, multiplication and division are learned relatively late and thus may be less practiced or well-learned than the operations of addition and subtraction. Second, division is the most difficult operation for students to learn and this difficulty may further hinder conceptual understanding involving division. Third, the operations of multiplication and division are themselves more conceptually complex compared to their addition and subtraction counterparts and thus the relationship between multiplication and division must be correspondingly more complex. Overall, the examination of multiplication and division inversion problems has significant potential for furthering our understanding of the development of children's conceptual knowledge in arithmetic.

Acknowledgments

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