Conceptual Change in Non-Euclidean Mathematics

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Abstract

To investigate the interaction of new information with deeply entrenched knowledge, we introduced participants to hyperbolic geometry, a form of non-Euclidean geometry. We trained participants through two different but mathematically equivalent forms: lines or figures. Participants who were trained on closed figures showed greater transfer than participants who were trained on lines. We gave participants different kinds of reminders at test to facilitate transfer. Explicit requests to apply training information to test items yielded no improvement, but presenting participants with relevant principles (but without information on how to apply those principles) greatly improved performance.

Keywords: Conceptual change; mathematics; non-Euclidean geometry.

Introduction

People don't learn a new domain of knowledge from scratch. Instead they have to integrate new information with their pre-existing beliefs, some of which may be false or inconsistent with the new knowledge. What happens when people confront information that conflicts with facts they hold true, especially when those facts are deeply entrenched?

This is the issue explored by research on conceptual change: the restructuring and perhaps abandonment of knowledge rather than the simple addition of new facts to a knowledge base. Most empirical work on conceptual change has focused on change in people's beliefs about scientific matters, such as children's concepts of animacy, models of the earth, and force (e.g., Carey, 1985; Vosniadou & Brewer, 1992; Ioannides & Vosniadou, 2001). Only recently have investigators pursued conceptual change in mathematics, including the development of the concept of fractions (Stafylidou & Vosniadou, 2004), rational numbers (Merenluoto & Lehtinen, 2004), negative numbers (Vlassis, 2004), and the illusion of linearity in geometry (Van Dooren et al., 2004).

Vosniadou and Verschaffel (1994) review several reasons why researchers and philosophers may have been reluctant to apply conceptual change theories to mathematics. One reason is that mathematics proceeds by deductive rather than empirical methods. Unlike explanatory models in physics or biology, which are developed and refined with the discovery of new data, coherent mathematical systems often do not depend on physical experience. While some mathematical concepts may be facilitated by sensory information, such as the relationship between Euclidean geometry and the observable world, other concepts are outside our experience (e.g., inaccessible cardinal numbers) or are even inconsistent with previously acquired knowledge (e.g., hyperbolic geometry). In addition, successive theories in math are not necessarily irreconcilable. In fact, as Corry (1993) points out, a new development in mathematics often does not lead to the rejection of the older theory but to a more generalized approach.

These factors make radical change in mathematics less salient than in science. Nevertheless, precisely because students expect mathematical knowledge to be unchanging, it can be especially difficult for them to encounter advanced math topics that force them to reconceive existing knowledge. This makes conceptual change in mathematics a particularly interesting area for study and a potential source of insight into how deeply entrenched knowledge interacts with new information.

Geometry lends itself well to this investigation. Hyperbolic geometry, a form of non-Euclidean geometry, is an interesting target for the study of conceptual change because of its conceptual similarities and dissimilarities to Euclidean geometry. In fact, the axioms of the two are identical, with one major exception: the replacement of the parallel postulate with the hyperbolic postulate. While in Euclidean geometry a line and an external point define a unique pair of non-intersecting lines, in hyperbolic geometry the two objects define an infinite set of non-intersecting pairs of lines. That is, given a line l and a point P not on that line, there are an infinite number of lines through P that do not intersect line l (rather than just one, as in the case of Euclidean geometry), as shown on the pseudosphere model in Figure 1.

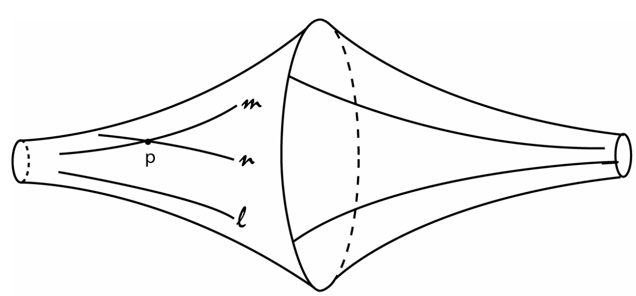

Figure 1. Hyperbolic parallel lines.

This property has important consequences for the mathematical objects and theorems in hyperbolic geometry and, indeed, for the very plane in which the geometry lies. While many geometric theorems are true in both Euclidean and hyperbolic geometries, others change dramatically. For example, in hyperbolic geometry, the interior angles of a triangle must add to less than 180°. Similarly, hyperbolic rectangles (quadrilaterals with four right angles) do not exist.

An understanding of both Euclidean and non-Euclidean geometry produces a different perspective on geometry as a whole. Hyperbolic geometry triggered a major change in philosophy, mathematics, and science in the nineteenth century. Following the realization that Euclidean geometry was not the only logically consistent geometry, mathematics evolved into an exploration of formal, logically consistent systems rather than systems that directly served the needs of science. It is therefore a real-world example of conceptual change in mathematics, a change that succeeded despite the counterintuitive nature of its conclusions. The question of how people manage to understand such a system seems especially pressing in view of evidence that basic geometric concepts are universal. Recent work by Dehaene, Izard, Pica & Spelke (2006) with an indigenous Amazonian group suggests that geometric concepts, such as points, lines, and parallelism, are core geometric intuitions available to all humans, regardless of formal instruction, accessibility of geometric terms in language, or experience with maps.

In the current studies, we were interested in how well people who had been exposed to Euclidean geometry through formal instruction in secondary school learned information about hyperbolic geometry. Developmental research has shown that children tolerate contradictory information as they learn (e.g., to integrate information that the Earth is round and their experience that is it flat, children construct a model of the Earth as a hollow sphere in which we live; Vosniadou & Brewer, 1992). But such inconsistent representations would render a mathematical system useless.

One might think that, since both Euclidean and hyperbolic geometries share most of the same postulates, the transfer between them would go fairly smoothly. However, small changes in core principles can also lead to interference between the systems, slowing down students' progress.

The present studies compare ways of conveying non-Euclidean information to see which methods facilitate or impede conceptual change.

Experiment 1: Figure vs. Line Training

We designed the first study to investigate whether emphasizing the holistic properties of the system or local building blocks better facilitated an understanding of hyperbolic geometry. While holistic information about the system may allow an individual to understand the relationships between different elements, it is also likely to interfere with previously acquired knowledge. On the other hand, although learning basic, more elemental information may provide a stronger foundation with less overall interference, it also has fewer explicit connections to other elements. Although the brief training participants receive is probably not sufficient to produce full-blown conceptual change, it can nevertheless provide an indication of the nature of the obstacles to such change.

In Study 1, participants learned hyperbolic information using either information about lines or closed figures. We wanted to see which type of information would lead to better generalization and transfer to hyperbolic objects not seen in training. Given that closed figures are composed of lines, it might be reasonable to expect that participants who receive hyperbolic training on figures would be better able to apply that information to lines than vice versa. If the closed figures provide information about their constituent elements, including lines, and make the relationships between the elements more accessible, then figure-trained participants solving line problems should outperform line-trained participants on figure problems. This is much like Larkin & Simon's (1987) conclusion that a diagram groups information in a way that facilitates processing and problem-solving. Similarly, closed figures (as compared to lines) may group conceptual information in a way that enhances accessibility and abstraction.

In contrast, lines can be seen as building blocks for closed figures. If the figure training condition requires participants to break down the analysis into lines and then synthesize it back into figures, we would expect the line condition to promote generalization more easily.

The hyperbolic postulate—the postulate which differs from the parallel postulate in Euclidean geometry—can be instantiated in a variety of mathematically equivalent forms, as seen in Table 1.

Table 1. Hyperbolic postulate equivalents.

Line instantiations
Given a line L and a point A not on that line, there are an infinite number of lines through point A that do not intersect line L.
If two non-intersecting lines are cut by a transversal, the alternate interior angles formed are unequal.
Closed figure instantiations
The interior angles of a triangle must add to less than 180 degrees.
If a quadrilateral has at least three right angles, the diagonals cannot bisect one another.

The top two postulates in the table refer to properties and relationships among lines, while the bottom two speak to properties of closed figures. We were able to use these mathematical equivalencies to construct two sets of training information: one about the properties of hyperbolic lines and a second about properties of hyperbolic closed figures (quadrilaterals and triangles).

Procedure In this computer-based task, participants first reviewed geometry terminology (e.g., *alternate interior* *angles, congruent*) that they would later see during training. After a pretest consisting of true/false questions to establish baseline Euclidean geometry knowledge, participants read one of two sets of hyperbolic training information, based on different but mathematically equivalent statements of the hyperbolic postulate, as in Table 1. In the line training condition, information was presented in terms of lines, while in the figure training condition, participants learned information about closed figures (e.g., triangles and quadrilaterals). The training information contained appropriate diagrams that had been constructed using a Poincaré disk, a model that represents hyperbolic geometry such that angle congruence has the usual Euclidean meaning (Greenberg, 1993). To control for the complexity of the diagrams, line-trained participants viewed diagrams in which the relevant lines appeared within closed figures, but these lines were highlighted on the screen.

Participants proceeded through the training information at their own pace, pressing the space bar to advance. After reading the training material, the participants received a test on the material, consisting of ten true/false questions. Although they received no feedback on the individual items, participants who missed one or more of the questions had to re-read the training materials. Training was repeated until they obtained a perfect score.

At posttest, all participants responded to the same forced-choice test items that they had seen in the pretest, concerning properties and relations of both lines and closed figures.

Materials The pre- and post-test items were identical in the two conditions and consisted of 45 true/false statements. Twenty of the statements referred to lines, twenty referred to closed figures. The remainder were filler items that referred to angles. The line and figure statements were constructed similarly and phrased such that half the statements of each type were true. For both types of statements, one half were *absolute* items—statements that had same truth value in hyperbolic and in Euclidean geometry (e.g., *Through any one point there exists an infinite number of lines that pass through the point*). The test items were presented on the screen as text statements. No diagrams were included with the test items. Absolute items were basic geometric principles that participants would have learned in secondary school geometry class. We could not expect that the undergraduate participants would come to the task free from all geometric knowledge, and both the question of interest and the practical considerations of the experiment required that the new information "piggyback" on the previously acquired geometric knowledge.

One half the test questions were *relative* items. That is, the truth values were different in the two geometries (e.g., *If lines A and B are a pair of non-intersecting lines, then any line which intersects A must also intersect B*). Both the relative items and the absolute items were phrased in such a way that the answers would be true for half the items in either geometry. In addition, some of the items were paraphrases of the training material or very closely related to it, while other items (hereafter, *transfer items*) required a number of inferences from the training information. While it was possible to construct proofs justifying the figure information from the line information (and vice versa), these proofs were not provided to participants, and they may sometimes be nontrivial (see the Appendix for an example).

Participants Sixty-seven Northwestern University undergraduates participated in the experiment, 34 in the line-training condition and 33 in the figure-training condition. The participants received partial course credit for an introductory psychology class.

Results and Discussion

To measure learning of the new hyperbolic information, we looked at performance on the transfer items. Not surprisingly, when participants were tested on their new hyperbolic knowledge, they did not perform as well as they did on the Euclidean pretest. Overall, posttest scores fell by 28 percentage points from pretest to posttest. In addition, participants performed best on the object type on which they had been trained. Participants in the figure condition were correct on 75% of figure items, but only 55% of line items. Participants in the line condition were correct on 68% of line items, but on 60% of figure items. Because the accuracy data are binary (either correct or incorrect), we performed a logistic regression to assess the effects of the independent variables, and we report the Wald test (Q_w) for these effects (Hosmer & Lemeshow, 1989). The interaction between

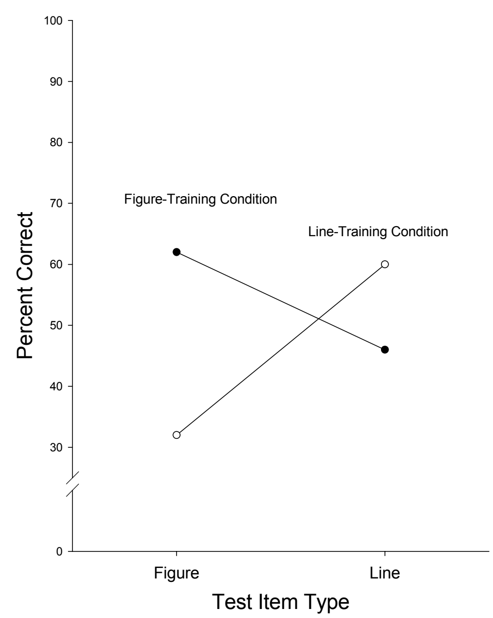

Figure 2. Posttest accuracy on hyperbolic transfer items, Experiment 1.

training condition and type of test item was significant in this analysis: $Q_w(1) = 18.41$, $p < .001$.

The key issue in this experiment is whether figure training or line training was more successful in conveying knowledge of hyperbolic geometry. Figure 2 shows the relevant data for the posttest items and indicates that participants in the figure-training condition performed better overall at posttest than did those in the line-training condition. The difference between training conditions is significant ($Q_w(1) = 4.29$, $p = .03$), as is the interaction between condition and test item ($Q_w(1) = 29.07$, $p < .001$). Accuracy for both groups was greater for the items on which they were trained. However, performance by the line-trained participants dropped to a below-chance score on the figure items, $t(135) = 4.57$, $p < .001$. This likely reflects a tendency to respond on the basis of prior Euclidean knowledge rather than the new hyperbolic information.

One interpretation of these results is that participants in the line-training condition were simply not learning the training information as well as those in the figure-training condition. Perhaps the line training was more difficult or confusing. However, participants in the two conditions achieved the same criterion-level performance during training and were about equally accurate during the posttest on the types of items for which they had been trained (line trainees on line items and figure trainees on figure items; see Figure 2). This makes it unlikely that participants in the line condition learned their training lesson less well than those in the figure condition. Another possibility is that the line information was more quickly forgotten, but since the test items immediately followed the final criterion test, this also appears unlikely.

Another straightforward explanation for poor performance is that participants in the line training condition simply didn't realize that the line information was relevant to the posttest figure items. In fact, we were struck by the number of participants who responded to our open-ended questions at the end of the experiment by saying that they hadn't been taught anything about figures or didn't realize that the line information was applicable. They may have learned the material well, but suffered from a kind of fixedness in the way they thought about the different types of objects (e.g., Duncker, 1945). That is, it is possible that—unless we are engaged in intentionally constructing geometric closed figures from lines—lines and figures appear to be completely different animals. Because figures may highlight relationships among their constituent entities, figure-trained participants may be less susceptible to this "objectification" of the items. If all that participants need is the insight that the line information should be applied to the closed-figure items, then an explicit reminder to apply hyperbolic line information to the figure items should produce improved transfer.

An alternative explanation is that, although participants in the line condition may have perceived the relevance of lines to figures, they weren't in a position to identify which line facts were appropriate when solving the figure problems. The figures themselves may have suggested irrelevant Euclidean information from prior knowledge of geometry. As we noted earlier, these participants' below-chance scores on figure items suggests this type of interference. For example, participants may understand the description *quadrilateral formed by two pairs of non-intersecting lines* simply as *parallelogram,* without thinking more deeply about the lines in the figure. Under these circumstances, we might expect that participants would have difficulty integrating the hyperbolic information into their knowledge of figures and would respond incorrectly with Euclidean answers. In contrast to line training, figure training may guide transfer of the abstract geometric information from figures to lines, allowing participants to apply the relevant relations to the test items. Just as it is easier to take apart a complex device into its components than to reconstruct it from those components, it may be easier to decompose knowledge of figures into knowledge of its component lines than to apply the reverse transformation. Decomposition may be a simple consequence of inherent part-whole relations; construction may require additional, explicit guidance.

If it is inherently more difficult to identify the relevant geometry information after hyperbolic line training, then simply reminding participants to use this information may not be enough. Instead, it may be necessary to provide explicitly the training information relevant to each figure item in order to make the connections between line and figure. This should facilitate use of the appropriate abstract geometric information and improve performance on the figure items at posttest.

By providing participants with different strengths of hints, Experiment 2 attempts to diagnose the difficulties with transfer from facts about hyperbolic lines.

Experiment 2: Hints during Test

Previous work in knowledge transfer has demonstrated that explicit reminders to use prior information may improve transfer, although with varying levels of success (e.g., Gick & Holyoak, 1980; Ross, 1984). To investigate the extent of the reminding necessary to improve generalization of line training to figure items, we decided to train participants on facts about hyperbolic lines and vary the instructions at posttest. In Experiment 1, we told participants to use the information they had just learned to answer the posttest items. In the current experiment, we tested what information would facilitate transfer.

If participants in the line-training condition simply didn't realize that the line information they had just learned could also be applied to the figure items—as many of them claimed—then a hint to relate the figures to the lines should improve performance at posttest. If, on the other hand, the difficulty lay in identifying or selecting the appropriate hyperbolic line-training information from what they had learned, then a mere hint to use the line information should not lead to transfer. However, explicitly reminding participants that a specific piece of line-training information is relevant may improve transfer to figure items.

Procedure and Materials The procedure and materials in Experiment 2 were similar to Experiment 1 with the following exceptions: 1) all participants saw hyperbolic line training materials and 2) participants saw one of three kinds of instructions immediately before the posttest items. The control group received the same instructions as participants in Experiment 1: They were told to respond to the items based on the geometry information they had just studied and the logical inferences from that information. The *hint* group received instructions emphasizing that, although they had not learned about figures, they should think about how figures were constructed from lines and, therefore, how the properties of hyperbolic lines would affect hyperbolic closed figures. The *specific reminder* group was presented with the same instructions as the hint group before posttest, but also saw 2–4 relevant statements from line training along with the relative test items. They were told that the information was relevant to the problem, but were not given any additional direction about how they should apply the line information. These "reminders" from the training information were statements that they had previously learned in order to pass criterion during training.

Participants Thirty-one Northwestern University undergraduates participated in the experiment in order to receive partial course credit in introductory psychology. There were 10 participants in the control condition, 10 in the hint condition, and 11 in the specific-reminder condition. None had participated in Experiment 1.

Results and Discussion

As in Experiment 1, participants' accuracy decreased from pretest to posttest, in this case by an average of 24 percentage points, $Q_w(1) = 40.76$, $p < .001$. Because all participants were trained on lines in this experiment, the decrease in performance was more pronounced for figure items than for line items, $Q_w(1) = 25.03$, $p < .001$.

The main point of interest is the effect of the reminders, and Figure 3 shows the relevant accuracy rates for the three groups of participants during the posttest. Participants who received specific reminders were about equally accurate on figure items as on lines, and they were the only group to achieve above-chance accuracy on figure items. This performance contrasts with that from the hint and the control groups, who were more accurate on lines than on figures. These groups apparently failed to transfer line knowledge to figure items. This difference between conditions produced a reliable interaction between hint type and item type, $Q_w(2) = 9.81$, $p = .007$. The control group's scores in the posttest were comparable to performance of the line-training group in Experiment 1 (compare the line-training condition in Figure 2). The hint group showed a trend toward improved performance on figure items, but this was not significant in our analysis, $Q_w(1) = 1.49$, $p = .22$.

The results from Experiment 2 suggest, then, that participants trained on hyperbolic line items need more than just a hint in order to apply the line information to the figure items. When participants are reminded of relevant line information in the context of the figure items, they appear better able to access and apply the relevant relationships between lines and figures. As we noted earlier, the inferences that participants needed to use this information were not necessarily easy. It is therefore of interest that simply naming the appropriate premises improved performance on figure items by about 40 percentage points, as Figure 3 shows.

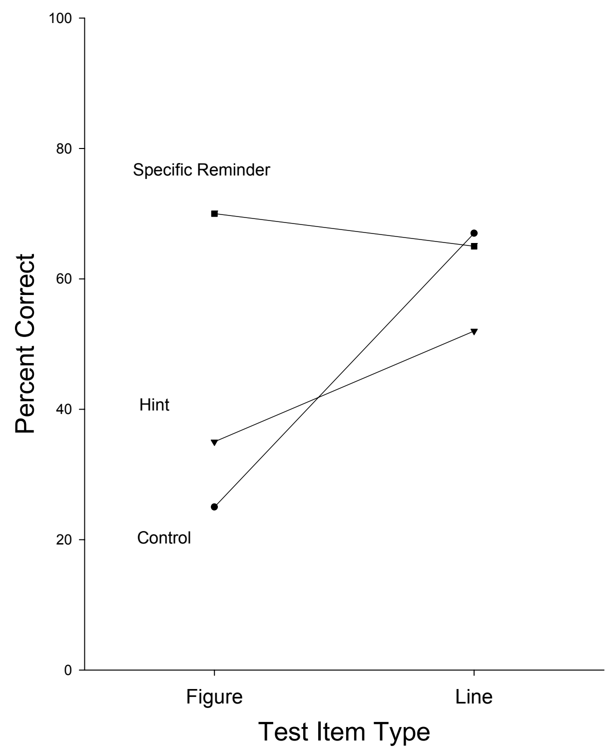

Figure 3. Posttest scores on transfer items, Experiment 2.

General Discussion

Everything we learn must be integrated into our existing knowledge. How this integration proceeds is a central question in the fields of concepts and problem solving.

In the current experiments, we looked at knowledge change in learning non-Euclidean geometry. We found that participants who received geometry training in terms of lines showed less transfer of knowledge than did participants who learned the information in terms of closed figures. We suggest that the figure training may have provided holistic information about line relationships, leading to advantages in applying geometric relations from figures to lines. In turn, this resulted in asymmetric transfer for the two training conditions in Experiment 1.

Even though some participants in the line-training condition reported they simply "hadn't realized" that they should apply the line information to figures, explicit instructions to do so in a follow-up experiment did not reliably improve transfer. Presenting participants with relevant training information at posttest, however, did significantly improve transfer from lines to figures. This further supports the idea that differences in the ease with which people can identify relevant information may be underlying the performance differences that we found in the first experiment.

The current studies reveal intriguing patterns in the way people incorporate new, possibly counterintuitive, mathematical information into their existing knowledge. Despite the additional complexity and processing requirements of hyperbolic geometric figures, participants seem to benefit more from encoding the new information in the form of figures than in the form of simple lines. For these purposes, more structured, holistic input seemed to be superior to training that focused on more specific building blocks. Although the highlighting of relational information seems a promising avenue for explaining these effects, future research should explore these issues more deeply. For example, both the extent of processing at the point of problem solving, as well as differences in Euclidean interferences would be candidate phenomena for study.

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Appendix. Example of informal proof

If we know information only about hyperbolic lines but need to determine how many right angles may be present in a hyperbolic quadrilateral, we may start with the fact that there can be at most one common perpendicular between any two nonintersecting lines. We can then form a closed figure by using the two nonintersecting lines as the base and summit of the quadrilateral, the common perpendicular as one side, and then drawing a second line segment between the two nonintersecting lines. However, the last segment may be constructed at a right angle to only one of the original non-intersecting lines, or the two non-intersecting lines will have more than one perpendicular in. common. As a result, the hyperbolic quadrilateral may have at most three right angles.

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