

Qualitative Spatial Reasoning: A Cognitive and Computational Approach.

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Abstract

In recent years a lot of psychological research efforts have been made in analyzing human spatial reasoning. Psychologists have implicitly used few spatial cognitive models, i.e. models of how humans conceptualize spatial information and reason about it. But only little effort has been put into the task of identifying from an algorithmic point of view the control mechanism and complexity involved in spatial relational reasoning. In this paper we extend the SRM model (Ragni, Knauff, & Nebel, 2005; Ragni & Steffenhagen, 2007) by new specifications and formalization of Baddeleys Working memory model. By the resulting model *CRQS* we are able to explain a number of new psychological effects of spatial representation and reasoning by the number of mental operations involved in solving these tasks. The discussion includes consequences of the formalization for the role of the central executive in spatial relational reasoning.

Keywords: Spatial Reasoning; Computational Modelling

Introduction

The ability to deal with spatial and temporal information is one of the most fundamental skills of any intelligent system and important in our everyday lives. When route descriptions are given, spatial information is usually contained in the description. While in engineering or physics it is most common to represent spatial information quantitatively, e.g. using coordinate systems, human communication mainly uses a qualitative description, which specifies qualitative relationships between spatial entities. But how is this information processed? Where is the focus of cognitive attention in processing qualitative information? In the following we focus on relational reasoning problems, e.g.

- (1) The red car is to the left of the yellow car.
The yellow car is to the left of the orange car.
The yellow car is to the left of the green car.
The green car is to the left of the blue car.

Is the blue car (necessarily) to the right of the orange car?

The statements are called *premises*, the cars are the *terms*, and the question refers to a putative *conclusion*. A premise of the form “The red car is to the left of the yellow car” consists of (two) objects, and a (usually binary) relation like “to the left of”. More precisely, the first object (red car) is the “to be localized object” (LO), which is placed according to the relation (left of) of the second object (yellow car¹), which is the “reference object” (RO) (Miller & Johnson-Laird, 1976).

¹In the following the objects are abbreviated by *R, Y, O, G* and *B*

Mental Models. For such relational reasoning problems, there exist several empirically validated effects in reasoning: the number of models (indeterminacy effect), the form of premises (figural effect), the wording of conclusion and the preference effect. These effects are explained by mental model theory (MMT) (Johnson-Laird & Byrne, 1991; Johnson-Laird, 2001). According to the MMT, linguistic processes are relevant to transfer the information from the premises into a spatial array and back again, but the reasoning process itself fully relies on model manipulation only. A *mental model* is an internal representation of objects and relations in spatial working memory, which matches the state of affairs given in the premises. The semantic theory of mental models is based on the mathematical definition of deduction, i.e. a propositional statement ϕ is a consequence of a set of premises \mathcal{P} , written $\mathcal{P} \models \phi$, if in each model \mathcal{A} of \mathcal{P} , the conclusion ϕ is true.

Without having an algorithmic formalization of a cognitive model, the task of testing and improving this model seems to be rather difficult, whereas the transfer of such cognitive models to AI systems seems to be even harder. Only a precise computational model, which defines parameters and operations, makes testable predictions. Furthermore, by using empirical data, formally specifying the role of the subsystems of a cognitive model, i.e. its store systems, it is possible to identify the necessary abilities of a computational model.

In this paper we formalize and analyze a combination of the preferred mental model theory and Baddeleys working memory model. Then we show how this model (i) is able to solve relational reasoning tasks, (ii) explains empirical results in the literature by the number of mental operations, (iii) report an experiment, which tests a new prediction made by the *CRQS*, and (iv) finally give an idea how the *CRQS* can help in specifying the role of the central executive (CE).

State of the Art

Psychological Background. For joining spatial reasoning and representation, it is necessary to specify and work out the main assumptions of mental model theory (MMT) and Baddeleys Working Memory Model (BWMM).

The mental model theory assumes that the human reasoning process consists of three distinct phases: The *model generation phase*, in which a first model is constructed out of

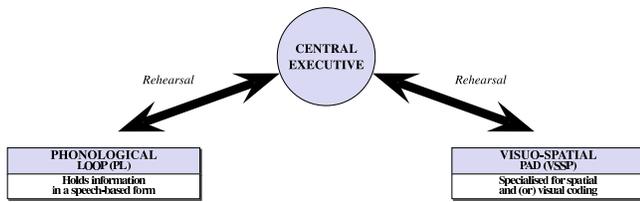


Figure 1: Baddeleys (1999) Working Memory Model

the premises, an *inspection phase*, in which the model is inspected to check if a putative conclusion is consistent with the current model. In the *validation phase*, finally, alternative models are generated from the premises that refute this putative conclusion. In our example presented above, the exact relation between “Y” and “G” is not specified and leads to multiple-model cases like R Y O G B or R Y G O B. This is caused by the indeterminacy effect and is mainly responsible for human difficulty in reasoning (Johnson-Laird, 2001). The classical MMT is not able to explain a phenomenon encountered in multiple-model cases, namely that humans generally tend to construct a *preferred mental model* (PMM). This model is easier to construct, less complex, and easier to maintain in working memory compared to all other possible models (Knauff, Rauh, Schlieder, & Strube, 1998). The principle of economicity is the determining factor in explaining human preferences (Manktelow, 1999). This principle also explains that a model is constructed incrementally from its premises. Such a model construction process saves working memory capacities because each bit of information is immediately processed and integrated into the model (Johnson-Laird & Byrne, 1991). In the model variation phase, this PMM is varied to find alternative interpretations of the premises (Rauh et al., 2005). From a formal point of view, however, this theory has not yet been formalized and is therefore not fully specified in terms of necessary operations to process such simple problems described above. In other words, the use, construction, and inspection of mental models have been handled in a rather implicit and vague way (Johnson-Laird, 2001; Baguley & Payne, 1999; Vandierendonck, Dierckx, & Vooght, 2004).

BWMM assumes a central executive, which is responsible for monitoring and coordinating the operations of two subsystems, the *phonological loop* (PL) and the *visuo-spatial sketchpad* (VSSP) (Fig. 1). The first subsystem, the phonological loop, stores information in a language-based form. The second subsystem, the visuo-spatial sketchpad, which is independent from the PL in terms of limits, stores visual and spatial information. Both subsystems are controlled by a Central Executive which is able to store and manipulate information in both subsystems. For combining the PMMT and BWMM, the following questions have to be answered: In which subsystem and how does the reasoning takes place? Which limits do the subsystems and the control process have? These questions are answered by results, which can be found in the literature: The deduction process in relational spa-

tial reasoning uses mental models (Byrne & Johnson-Laird, 1989). The mental models can be located in the WMM in the visuo-spatial sketchpad, where the construction and manipulation of the mental models can be located as well. The model in the VSSP is manipulated by a special device which is called focus. The PL uses some dynamic memory allocation like the first-in-first-out principle (Vandierendonck et al., 2004; Baguley & Payne, 1999).

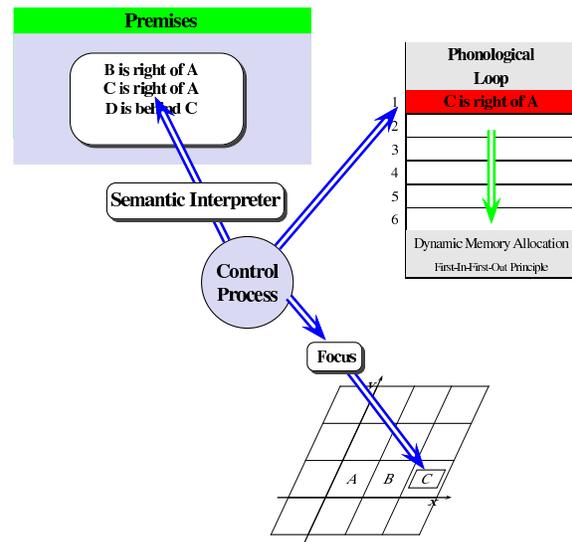


Figure 2: The CRQS-Model

Computational Models. A first computational model for mental models was presented 1991 (Johnson-Laird & Byrne, 1991). This model is mainly able to insert objects into an array and to generate a mental model. This model simulates and explains experimental results of three-term problems. The most general model in reasoning with relations is the UNICORE model (Bara, Bucciarelli, & Lombardo, 2001). This model is able to explain several effects on relational reasoning (e.g. figural effect), but so far it does not model the variation phase or include any representation of human memory. A first approach to specifying a computational model for the preferred mental model theory has been the SRM presented by (Ragni et al., 2005). This model consists of an input device for the premises, a two-dimensional spatial array in which the mental model is constructed, inspected, and varied, and a focus, which performs these operations. The application of a standard cost measure was able to explain empirical results. But this model still contains some limitations. Complex relations, defined by the number of dimension and sources of variations that are related (Halford1, 1998), are not definable. There is no working memory representation and the model variation phase is not specified.

The CRQS Model

Each computational model is based on assumptions and abstractions depending on its aim. The CRQS-Model (Cognitive Relational Operating System) which formalizes the

Table 1: The instruction set of the \mathcal{CRQS} .

Control process operations	
readnext()	read the next premise from SI and save values to the variables LO, RO, and REL
SubSystem(sys)	change sub system the central process is working on; sys can be the PL or the SA
Control Flow	
if <i>val</i> then {instr. block} [else {instr. block }] while <i>val</i> do {instr. block}	test whether <i>val</i> is true and process first instruction block else 2nd block is processed process instr. block as long as <i>val</i> <> 0
Operations on Phonological Loop	
writep(prem)	write premise into loop
annotate(<i>o,a</i>)	annotate object <i>o</i> with <i>a</i>
annotations(<i>o</i>)	return annotations of objects <i>o</i>
annotated(<i>o</i>)	return true if <i>o</i> is annotated
Focus Operations	
fmove(<i>d</i>)	move the focus to direction <i>d</i>
fread()	read cell where focus is on, return false if cell is empty
fwrite(<i>o</i>)	write object <i>o</i> to cell where focus is on
newLayer()	create new empty layer
Complex Sub-Programs	
shift(<i>o, d</i>)	shift object <i>o</i> to direction of <i>d</i>
exchange(<i>o,rel,concl</i>)	exchange object <i>o</i> to direction of <i>rel</i> generating a new model
fmove(<i>o</i>)	move focus to object <i>o</i>
inverse(<i>rel</i>)	compute inverse relation to <i>rel</i>
layer(<i>o</i>)	returns the layer of object <i>o</i> ,
merge(<i>l₁, l₂</i>)	merge layer <i>l₁</i> and <i>l₂</i>

WMM and PMMT consists of: A conceptualization of the WMM (with subsystems), a *manipulation device* for the mental models, a (relational) language describing object positions, and a *semantic interpreter*, interpreting the language. The central place where models are located is the VSSP. The VSSP is a spatial array (SA) of two-dimensional grids, called *layer*, in which the models are generated and manipulated by a device called *focus*. The focus can perform a small number of operations like moving, reading, and inserting. For example for 'A left B' and 'C right D', there are two possible submodels, each placed in its own layer, so that submodel *AB* would be in the first and *CD* in the second layer. Formally, the \mathcal{CRQS} is a 6-tuple (I, SI, A, F, PL, C) with:

I: the input device

SI: the semantic interpreter; interprets the input of *I*.

A: a spatial array containing the layers. We set: $\omega(A)$ the objects held by array *A*; $\lambda(O)$: the layer of object *O*.

F: the focus working on the spatial array, able to perform move (L,R,B,F,No-Move), group, and shift operations.

PL: the PL, a memory for storing verbal information.

C: a control process, which uses the instructions of Fig 1.

Problems related to the ambiguity of spatial relations are not accounted. The model interprets the string "A is left of B" as: both objects are in the same line and A is to the left of B. The relations "right", "front", and "behind" are equivalently defined. When processing natural language strings, the

meaning of the input has to be interpreted. In linguistics, as well as in psychology, the existence of a *semantic interpreter* (SI) is assumed, which in our model maps syntactically analyzed texts to the formal representation. The semantic interpretation is not part of this paper. We simply assume a parser that provides the correct interpretations to the system. If indeterminacy occurs, information about other possible models must be stored. Since a mental model is only a representation, information of other models must be held in another subsystem. This information is psychologically modeled by annotating objects (Vandierendonck et al., 2004). Since we do not know how humans encode such information, we use the full premise as annotation. The appropriate memory system in the WMM for this kind of propositional information is the PL. This is consistent with neuropsychological evidence (Knauff, Mulack, Kassubek, Salih, & Greenlee, 2002). The PL is managed by a dynamic memory allocation system like FiFo or least-recently-used strategy (LRU), which allows the modeling of activated objects.

Since both systems, the SA and the PL, are only memory systems and the focus manipulates only the SA, a *control process*, which manages the \mathcal{CRQS} , is needed, managing the subsystems and controls the focus operations on the SA. The control process has a limited instruction set (Fig. 1). Several instructions directly control read/insert/move operations of the focus, statements to branch or loop the control flow, and simple test instructions. With this set of instructions, algorithms for all three reasoning phases can be defined and different insertion strategies can be tested and compared. The premises are read and interpreted iteratively by the SI, and the control process inserts the new encountered information immediately into the model by moving the focus in the SA and adding indeterminacy information to the PL. The focus has the ability to create new layers for premises that cannot be constructed into one layer.

In the following we present the algorithms for the construction, inspection and variation on the basis of problem (1) with abbreviated initial letters for the car objects.

Model construction The algorithm for the model construction has to distinguish five types of premises (O_1, r, O_2) to place the objects of the premises: (1) $\omega(A) = \emptyset$ (first premise), (2) $O_1 \in \omega(A)$ and $O_2 \notin \omega(A)$ or vice versa, (3) $O_1, O_2 \notin \omega(A)$, (4) $\lambda(O_1) \neq \lambda(O_2)$ (connecting two layers), (5) $\lambda(O_1) = \lambda(O_2)$ (additional knowledge).

The construction process begins with the first premise and an empty layer. First the RO is placed, then the focus moves in the direction of the relation and places the LO to the next free cell. In our example *Y* is inserted first, the focus moves to the left and inserts *R*. The algorithm (Fig. 3) checks the type of each new premise and inserts the object(s) according to the specific case. For premises of type 2 only one object will be inserted into the model according to the already contained object. If the new object cannot be placed as a direct neighbor, the model structure is indeterminate, so the control process annotates the object by inserting the relational

information as a proposition into the PL, and the focus places the present object according to the fff-principle (first free fit). For premises of type 3, where neither of both objects are contained in the model, a new layer is generated, and both objects will be placed as seen in the beginning of the model construction. If both objects are contained in different layers (type 4), both layers have to be merged according to the relation of the premise. Premises of type 5 specify additional knowledge for two objects contained in the same layer. They are processed by a model variation step, trying to check if the inverse premise holds in all variations of the actual model. If a counter-example exists, it is a model containing the additional knowledge. The second and third premise are of type 2 because *Y* is already in the model, so *O* and *G* are inserted to the right of *Y* according to the fff-principle. Because *G* cannot be placed adjacent to *Y* it is annotated with 'right *Y*'. The next processed premise is also of type 2 and object *B*, that is not in the model, is inserted directly to the right of *G*. Because *G* is annotated, *B* has to be annotated too. Now the construction phase is complete and the resulting model is shown in the first line of Fig. 6.

```
def constructModel():
    readnext()
    fwrite(RO)
    fmove(inverse(REL))
    fwrite(LO)
    while readnext() do
    { if type2 then
      { fmove focus to contained obj
        fmove focus
        while not placed do
        { if fread() then
          annotate missing obj
        else
          fwrite missing object
          placed = true} }
      if type3 then
      { l = newLayer()
        fwrite(LO); fmove(inverse(REL))
        fwrite(RO) }
      if type4 then
        merge(layer(LO), layer(RO))
      if type5 then
      { newModel=valConcl(LO, inverse(REL), RO)
        if newModel then
          writeModel() }}
```

Figure 3: The construction algorithm.

Model inspection After model the construction, the *inspection phase* searches for new information (cp. Fig. 4) that was not specified by the premises. The focus moves to the first given object (RO) and from there it inspects the model according to the relation in order to find the second object (LO). The search process terminates after $O(n)$ steps, since the model is bounded by the number of objects.

Model variation The *model variation* comes into play if a conclusion must be verified or if additional knowledge of two already contained objects must be processed during the model construction process. The focus starts in the variation process with the PMM and varies the model with local transformations to generate a counter-example to the putative conclusion. The variation process starts from the generated

```
focus_inspect(LO, rel, RO):
    fmoveto(RO)
    while fread != LO do
    { fmove(rel)
      if fread()==LO then
        return true }
    return false
```

Figure 4: Pseudo code for the inspection algorithm.

PMM (in which the putative conclusion holds). The algorithm checks if one of the objects in the conclusion is annotated. Annotations on objects specify the positional relation to reference objects, which we refer to as *anchor*. If the annotations on one of the objects include the relation and the other object of the putative conclusion then the putative conclusion holds. The same argument holds if none of the conclusions' objects is annotated because then the positions of the objects are determined. If there is an annotation only one of the objects, as in the example conclusion '*B* is to the right of *O*' (see Fig. 6), the only object of the conclusion which is to be moved is *B* and not *O*. This goes along with the use of annotations, i.e. in the construction process an annotation is created only for indeterminate object positions. If the object which is to be moved has an anchor, it may be necessary to move the anchor first. To provide an example: *B* cannot be moved because *G*, the anchor of *B*, is a direct neighbor of *B*. Thus, the algorithm first exchanges the anchor to the left of *O*, which is possible since *G* is the anchor of *Y*. Now the counter-example can be generated by exchanging *B* beyond *O* because the anchor of *B* can be placed to the left of *O*, so false is returned. If both objects are annotated, then first the LO of the putative conclusion is exchanged. LO is moved into the direction of RO until its anchor is reached. If this results in the generation of an inconsistent model, the algorithm stops, and returns false. It is possible that the anchor object is between LO and RO, so LO is exchanged until it reaches the anchor. Then the anchor object is exchanged recursively towards the RO. If no further exchanges to RO are possible, the exchange process starts to exchange the RO into the direction of LO.

Psychological Results

There are several psychological experiments investigating spatial relational reasoning. The findings vary from the classical question determinacy/indeterminacy (Byrne & Johnson-Laird, 1989), the wording of conclusion to questions which investigate the role of relational complexity. What do all these different effects have in common? The measure of effects can be explained by one general concept: The number of "mental operations" necessary to perform the tasks is sufficient to explain the results. We analyze the role of the Central Executive w.r.t relational complexity and the model variation phase.

Relational Complexity and Operations. The influence of relational complexity in the model generation phase is studied by (Goodwin & Johnson-Laird, 2005). Participants had to infer the relative starting positions of 4 runners on 5 lanes in a race, given by:

```

validateConclusion(Model, concl):
{ if layer(LO) != layer(RO)
  return false
  if not check(concl)
    return false
  if conclusion in annotations(Model)
  or inverse(concl) in annotations(Model)
    return true
  if LO not in objects(annotations)
  or RO not in objects(annotations)
    return true
  if not exchange(RO, relation, concl)
    return false
  else
    if not exchange(LO, rev(relation), concl)
      return false
    return true }

```

Figure 5: Model variation algorithm. The exchange method exchanges an object according to the given premise and conclusion to find a counter-example. The object is exchanged until the 'anchor' object is reached, from there it recursively proceeds with the anchor and so on.

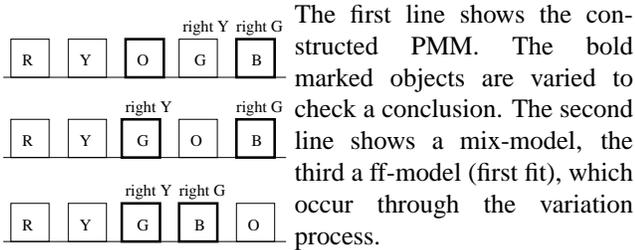


Figure 6: The variation process.

- (2) A is left of C and B is left of A
 B is left of C and D is left of B
 A is further away from C than B is from A
 Who is closer to the empty lane, B or A?

which results in the model: D B A C
 The third premise yields the allocation to the lanes. They tested two kinds of the third premise: (i) A is further away from C than B is from D. (ii) B is further away from C than D is from A. The third premise yields the allocation to the lanes. The answer in both cases is that A is closer to the empty lane than B. The difficulty has been explained as the number of individuals to be simultaneously integrated (hence (i) is easier than (ii)). The authors introduced the *principle of integration* to explain this effect. But is this really necessary? The result can be explained in terms of the number of necessary operations. We show the computations of the *CRQS* (for (ii)): First the distance between D and A (with the focus) is measured and stored (dist.:2). Then the focus searches for object B and then counts the steps to get from B to C. After that the focus moves object C one position to the right. The focus needs 5 operations more for (ii) than for (i).

Testing the *CRQS*. How does the variation phase work? The *CRQS* predicts a continuous transformation which starts from the PMM. This prediction is now tested empirically.

Material, Procedure, and Participants. We designed 20 indeterminate 5-term series problems with 4 premises of

problem (1) with relations *left of* or *above*. They consisted of fruits (kiwi, mango, pear, apple, peach). The participants read the four premises self-paced in a sequential order. Each premise was displayed in the center of the screen and disappeared before the next premise was shown. For each problem one of six models (written out fruit names) was presented, three valid and three invalid one. The participants were asked to decide whether the offered model was a consistent model of the previous premises. The models were designed in different ways: (1) to (3) are consistent models, which are con-

(1)	R Y O G B	(4)	R O Y G B
(2)	R Y G O B	(5)	R G O Y B
(3)	R Y G B O	(6)	B Y O G R

Figure 7: Models presented in experiment.

structed corresponding to (1) the fff-principle (PMM), (2) a mixture of the fff- and the ff-principles, and (3) the ff-principle. The models (4) to (6) are inconsistent models with an exchange in (4) the second and third term (near exchange), (5) in the second and fourth term (middle exchange) and (6) in the first and last term (far exchange). 21 participants from the University of Freiburg took part in the experiment.

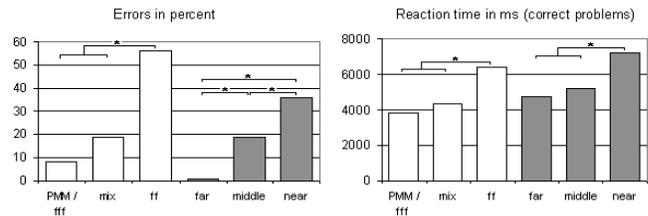


Figure 8: Experimental results of 5-term series problems.

Results and Discussion. We analyzed the accuracy and reaction times for the offered models. We summarize only the main findings, for a full report see (Ragni, Fangmeier, Webber, & Knauff, 2006b). The participants which had to validate models congruent with the fff-principle produced the highest number of correct answers (92%) and the reaction times were the smallest ($M = 3797$ ms, $SD = 1640$ ms). There was a decrease in both accuracy and reaction times from those models constructed with the fff-principle to the mixed models to ff-models (cp. Fig. 8). From the construction point of view the fff- or the ff-principle makes much more sense with regard to simplicity and use of a single strategy compared to the mix-principle. For the invalid models there was an increase in the error rates from far to near exchange and the reaction times were also longer for far and middle. As in (Ragni, Fangmeier, Webber, & Knauff, 2006a) the results indicate that participants more likely construct a PMM (fff-principle), because accuracy and reaction times were significantly better if they were asked to validate a PMM. The results indicate that participants construct first the PMM out of the premises (following the fff-principle), and since it is easier to construct, than

the mix-model and than the ff-model, the accuracy and reaction times were better in the mixed case than in the ff-case. The results of the *CRQS* are confirmed: less operations are necessary to transform the PMM into the mix-model than to transform the PMM into the ff-model (cp. Fig. 6).

General Discussion

The preferred mental model theory as well as Baddeleys WMM can explain several of empirical results in spatial reasoning. But both theories have not yet been brought together or been formalized. Since human reasoning is based on mental operations as well as on mental structure, only a cognitive as well as formal model, comprising both aspects, is able to explain intra- and inter-individual differences. This is the starting point for our investigation and formalization of both theories. By benchmarking the resulting cognitive model on experimental results from the literature we could show that the *CRQS*² is able to cover a wide span of effects, from preference effects to relational complexity effects (Ragni & Steffenhagen, 2007). More important, several *CRQS*-predictions like the insertion principle (fff-principle) and the continuous transformation process have been empirically validated in human experiments (Ragni & Steffenhagen, 2007). Our symbolic approach has not yet covered the property that objects can have different activation levels. It seems reasonable to import the activation function from ACT-R. Can we determine, starting from the formalization of the data structure and for a given (solvable) problem, computational properties of the Central Executive? Take for instance problem (2) (Ragni et al., 2006a). How is the query *A as near to D as C is near to E* to be processed and where is this information stored? Participants explained that they used a divide-and-conquer strategy to solve this problem. That first the distance between A and D is determined and than the distance between C and E and then both distances have to be compared. Numeric information is generally not assumed to be stored in the VSSP, and since the PL is a mere rehearsal process, where information can be stored but not manipulated, the memory system in which this information is stored and manipulated remains unclear. It seems reasonable that in the Central Executive two operating cells in which incremental operations (+1) and comparisons (<, =, >) take place can be assumed. This existence is implicitly assumed in (Lemair, P. and Abdi, H. and Fayol, M., 1996). Such implications of an algorithmic formalization of BWMM are to be investigated next.

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²Further information and an implementation can be found at <http://gkiweb.informatik.uni-freiburg.de/~srm>

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