Using Gestalt Principles to Compute Analogies of Geometric Figures

Angela Schwering, Ulf Krumnack, Kai-Uwe Kühnberger, Helmar Gust
(aschweri | krumnack | kkuehnbe | hgust@uos.de)
Institute for Cognitive Science, Albrechtstr. 28
Osnabrück, 49076, Germany

Abstract

Analogy making is a highly sophisticated cognitive process in which two stimuli are compared and analyzed for common patterns. Analogical reasoning is the adaptation of knowledge about one stimulus such that it can be applied to the second stimulus and new analogous inferences can be drawn. Analogy making requires intelligence, since such analogous patterns and transfers often are not obvious and require a certain conceptualization of the stimuli. Intelligence tests often use proportional analogies, where subjects have to find a follow-up for a series of geometric figures.

Heuristic-Driven-Theory-Projection (HDTP) is a formal system for analogy making and analogical reasoning. This paper shows how HDTP can be used to analyze proportional analogies between geometric figures. It uses VREG, a language of perception developed by Dastani (1998), to compute different Gestalts from 2-dimensional geometric figures. Depending on the Gestalt perception, HDTP proposes analogical inferences at different levels of cognitive plausibility.

Since HDTP describes stimuli formally in first-order logic, it is possible to separate the description of a concrete stimulus from the specification of general Gestalt principles of perception. Therefore it is possible to generate conceptualizations of geometric figures on the fly (called re-representation). The ability of automatic re-representation is one of the central advantages of HDTP compared to other analogy models.

Keywords: analogy making, proportional analogy, analogical reasoning, Gestalt perception of geometric figures

Introduction

Analogy making is a high-level cognitive process of transferring knowledge from a source domain to explain unknown phenomena or solve novel problems in a new situation (the target domain). The difficulty of analogy making is identifying parallels between the different elements, e.g. find two elements which play comparable roles in the overall structure.

Proportional analogies have the general form (A:B)::(C:?) where A, B and C are given elements and the question is which element is suitable for "?" in order to have the same structural relations between the source elements A and B as between the target elements C and "?". While analogies are often built across *different* domains, in proportional analogies the source and the target elements typically come from the same domain. Proportional analogies were studied in the natural-language domain (Indurkhya; 1989, 1992), in the string domain (Hofstadter & Mitchell; 1995) and with respect to geometric figures (Dastani; 1998, Dastani & Indurkhya; 2001, Dastani, Indurkhya, & Scha; 1997, Evans; 1969, O'Hara; 1992, O'Hara & Indurkhya; 1995).

Figure 1 shows a proportional analogy problem between geometric figures as it is commonly used in intelligence tests. There are several geometric figures possible as solutions for this analogy problem, however they might not be equally plausible and often depend on the perception of the elements A and B.

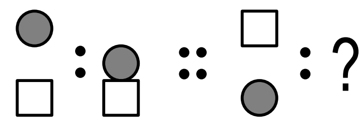


Figure 1: Proportional Analogy.

When the human sensory system observes a geometric figure, it transforms the unstructured information into a structured representation of a pattern or a set of objects. Human perception tends to follow a set of Gestalt principles of organization (Wertheimer; 1912).

A formal analogy model computing proportional analogies between geometric figures requires a good computational model for Gestalt perception. This paper shows how "Heuristic-Driven Theory Projection", a formal model for analogy making, can be used to compute proportional analogies of geometric figures formalized with Visual REGularities (VREG), a language to formalize different Gestalt perceptions of visual stimuli.

Heuristic-Driven Theory Projection (HDTP) is a computational approach for analogy making and analogical reasoning. It represents the source and the target stimulus via two formal theories based on first-order logic. In the analogy identification process, HDTP compares both theories for common patterns using anti-unification: The formulas of both theories are compared to find a generalization subsuming both, the source and the target formula. The result is a generalized theory describing formally the common patterns of the source and the target stimulus and the analogical relation between them.

The formal basis of HDTP allows not only the representation of the geometric figure, but also the general description of the Gestalt principles of perception. The separation of the knowledge about the stimulus and the abstract knowledge of how humans perceive Gestalts allows HDTP to compute different conceptualization of the same geometric figure on-the-fly based on different Gestalt principles. Like humans, HDTP proposes different possible analogies depending on the conceptualization of the figure.

The remainder of the paper is structured as follows: In the next section, we introduce VREG, the language of perception used to formally represent geometric figures. The third section explains the analogy making and analogical reasoning mechanisms of HDTP. In the fourth section we first introduce an example for a proportional analogy, which is used to demonstrate how HDTP computes analogies. We end by summarizing the paper and giving directions for future work.

Language of Perception

Dastani et al. (Dastani; 1998, Dastani & Indurkhya; 2001, Dastani et al.; 1997) developed a language of perception to describe two-dimensional geometric figures conceptualized following the principles of Gestalt psychology.

Gestalt Psychology

In human perception, the numerous pixels detected by the visual sensory system are restructured and composed to form coherent shapes and figures. The law of Prägnanz says, that humans tend to experience stimuli in a possibly good Gestalt way, i.e. as regular, simplistic, ordered or symmetrical as possible (Koffka; 1935, Köhler; 1929, Wertheimer; 1912, 1954). The Gestalt psychology identified different principles according to which humans construct Gestalts of geometric figures:

- The principle of proximity states that elements being spatially close together tend to organize into units (figure 2).
- According to the principle of good continuation humans tend to group elements together which follow one continuous direction (figure 3).
- Humans tend to perceive shapes as closed forms, e.g. four connected lines are perceived as a quadrangle rather than as separated lines (closure principle).
- According to the similarity principle, same and similar elements are perceived as groups.
- The principle of symmetry states that humans group symmetric figures together regardless of their distance.
- The principle of habit says that the perception of humans is based on their experience and humans prefer well-known Gestalts to new Gestalts.

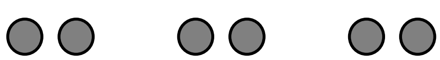


Figure 2: According to the principle of proximity humans tend to perceive three groups of two circles.

Applying the various Gestalt principles to the same figure might result in different perceptions: Although there have been identified certain rules which Gestalt principle is cognitively preferred (Wertheimer; 1912), there does not exist a fixed hierarchy of Gestalt principles. The perception of geometric figures may differ among humans and among contexts.

The principle of proximity and continuous direction will be important for our example analogy of geometric figures in the paper. Figure 2 and 3 show Gestalts based on these two principles: in figure 2, humans perceive three pairs of two circles rather than 6 separate circles. In figure 3 the grouping of circles to the two lines A-D and B-C is preferred to A-B and C-D, because the latter one involves a sharp turn.

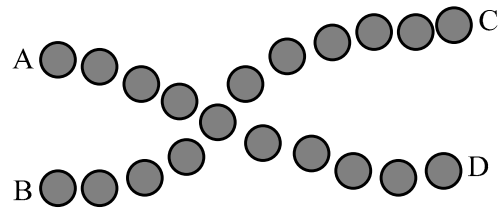
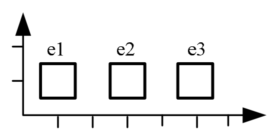


Figure 3: According to the principle of good continuation humans tend to perceive two lines of circles A-D and B-C.

A Language for Visual REGularities

The Visual REGularities language VREG is an algebraic language to describe two-dimensional figures. The representation of a geometric figure in VREG is an iterative process: it starts with a list of all primitive visual elements (figure 4 consists of three elements *e1*, *e2* and *e3*). Afterwards different transformations or functors can be used to describe elements by their relation to other elements: Instead of describing each element separately, the pattern in figure 4 can be described as an iteration of squares moved two units to the right along the x-axis.



$$Iter(e1, move^{i}_{xpos}, 3)$$
$$move^{i}_{xpos} = 2*(i-1)$$

Figure 4: Iteration of squares moved along the x-axis.

The language VREG is defined over the signature $< E, \Omega, \Psi, Iter, Sym, Comp, Unit >$:

- E is the set of primitive visual elements, e.g. the circles or the squares in the geometric figure.
- Ω is the set of parameterized transformations and Ψ is the set of reflection transformations, e.g. $move^{i}_{xpos}(i) = 2*(i-1)$ or $shape = \{(circle, square)\}$.
- $Iter, Sym, Comp, Unit$ are four functor names, corresponding to iteration, symmetry, composition and unit of visual patterns (Dastani; 1998, p. 62).

Dastani has proposed a measure of complexity which shall reflect the cognitive plausibility of the representation.

The computation of a cognitively plausible representation for a geometric figure is Gestalt-driven. Of course, no algorithm coding the human Gestalt perception exists. However, there are several approaches which approximate the different principles relatively well. The organization into units according to the proximity principle can be computed for example by the agglomerative hierarchical clustering (Dubes; 1993, Jain & Dubes; 1988): this algorithm calculates spatial distances between all primitive elements and groups the two nearest into one unit. This process is iteratively repeated with all remaining primitive elements and units. The optimal grouping in units is reached and the algorithm stops when the spatial distances exceed some threshold.

Similarly we can find algorithms to approximate other Gestalt principles: continuation can be computed by searching for element chains where the angle of two following chain sections is approximately 180°.

Heuristic-Driven Theory Projection

HDTP is an analogy model with a mathematical sound basis: The source and the target domain are represented by theories based on first-order logic. In the domain of geometric figures the formal representations of the source and the target figure reflect the human Gestalt perception. HDTP computes different possible analogical solutions, which are ranked according to their cognitive plausibility. HDTP distinguishes between domain knowledge—facts and laws holding for one specific domain only—and background knowledge, which is generally true for all domains. Concerning our application area of analogies between geometric figures, background knowledge provides a vocabulary to model geometric situations and rules to describe equivalence of different modelings, while domain knowledge describes only individual figures.

HDTP (Gust, Kühnberger, & Schmid; 2006) uses anti-unification to identify common patterns in the source and target. Anti-Unification (Gust, Kühnberger, & Schmid; 2003, Plotkin; 1970, 1971) is the process of comparing two formulae and identifying the most specific generalization subsuming both formulae. We use anti-unification to compare the source theory with the target theory and construct a common, general theory which subsumes a possibly great common structure of the source and the target domain. We illustrate the mapping process with an example from the geometric domain: the primitive visual elements in figure 5 can be expressed via different transformations: the transformations from the white circle to the white oval (marked with 1) is a move along the x-axis $move_{xpos}(x) = x + 4$ and a stretch $strech(s) = 2 * s$. On the target side we apply the same move transformation and a stretch transformation $stretch(s) = 1/2 * s$.

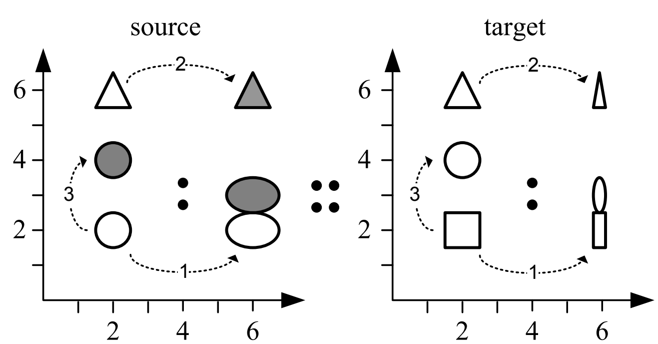


Figure 5: Anti-unification of functors in geometric figures.

HDTP anti-unifies both move transformations and both stretch transformations: since the move transformations are identical they need no generalization. The stretch transformation differs in the parameters which are represented by a variable at the general level (figure 6).

$$\begin{tikzcd} & strech(s) = X * s \arrow[dl, "X \leftarrow 1/2"'] \arrow[dr, "X \leftarrow 2"] & \\ stretch(s) = 1/2 * s & & strech(s) = 2 * s \end{tikzcd}$$

Figure 6: Anti-unification of two transformations.

The transformations 2 in figure 5 are a colour change $colour = \{(white, grey)\}$ in the source domain and a $stretch(s) = 1/2 * s$ in the target domain. Two different transformations are aligned via anti-unification with a generalized transformation *T*. Not all transformations in the source domain have to align to a transformation in the target domain: the colour change $colour = \{(white, grey)\}$ marked as transformation 3 in figure 5 does not have a corresponding transformation on the source side.

HDTP also compares and aligns the primitive visual elements: in the example, both circles could be aligned. In this case the generalization would be again a circle. Another possibility would be to align the circle and the square which results in a generalization to a generic visual element. The selection process is determined by the heuristics of HDTP which can incorporate Gestalt principles in the case of the geometric domain.

As mentioned above, HDTP is able to represent factual knowledge about the domain—e.g. the shape and colour of elements—but also to represent abstract coherences such as transformation rules or rules for Gestalt perception. Being a formal model, HDTP can automatically compute inferences from the theories describing the domains: In the context of geometric figures, this means that HDTP can compute new representations of the figure on-the-fly and automatically propose different Gestalts perceptions. The ability of automatic re-representation is considered to be one of the main advantages of logic-based analogy models: Usually, domains are not represented in advance in a way that the analogy can be perceived immediately. Analogies often have a very specific perspective and the conceptualization is highly context dependent. The conceptualization of a domain must be modified and adapted to make analogous patterns to another domain obvious. Logically consistent inferences allow for re-representation without changing the semantics.

Proportional Analogies of geometric figures

This section introduces the running example with different solutions to the proportional analogy. Afterwards it explains how HDTP computes these analogies.

Running Example

Depending on the Gestalt perception of the geometric figure, humans complete the same proportional analogy differently. However, usually you can identify one or two cognitively preferred solutions.

Figure 7 shows one potential analogy: according to the proximity principle, the square *s1* is grouped to the circle *s2* (unit1) and *s3* is grouped to *s4* (unit2)[1]. The second unit is an iteration of the first unit with the circle moved towards the square.

Having this perception of the geometric figure of the source domain, the analogy is completed by repeating the "iteration and move" transformation in the target domain: the target domain repeats the same figure upside down.

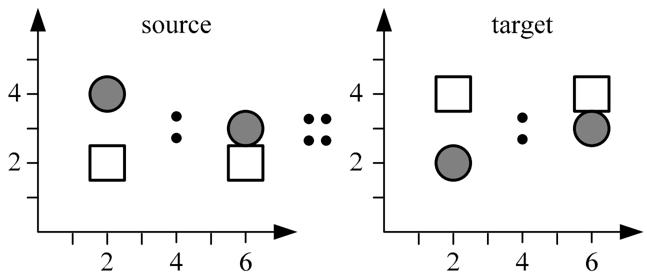


Figure 7: According to the law of proximity, the circle and the square below are grouped together. The analogy is complete by repeating the source figure upside down.

Figure 8 proposes a different analogy for the same geometric figure: according to the principle of similarity, the squares form one unit and the circles another. While the squares have the same height level, the circles form a line with a downwards direction. Following the principle of good continuation, the analogy is completed by a square *t3* at the same height level and a circle *t3* lower than circle *t1*.

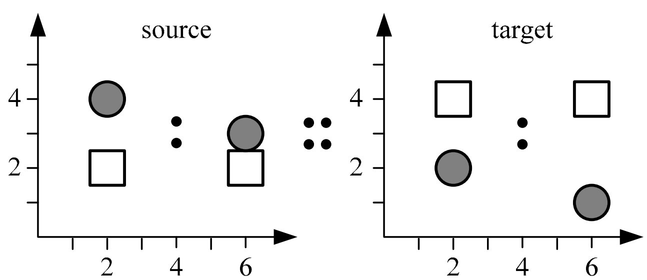


Figure 8: According to the principle of good continuation, the circles are grouped together and the squares. The analogous picture is created by transferring the continuous downward direction of the circles.

Our pre-tests provided an indication that the solution shown in figure 7 was slightly preferred to the solution in figure 8. The analogy in figure 11 was hardly ever mentioned. The experimental investigation of cognitive plausibility of analogies, however, is subject to future work.

[1] The visual elements on the source side are numbered by s1 to s4 bottom-up and left-right (the square at (2,2) is *s1*, the circle at (2,4) is *s2*, the square at (6,2) is *s3* and the circle at (6,3) is *s4*. The elements on the target side are numbered by *t1* to *t4* respectively.

Analogy between geometric figures in HDTP

For the investigation of proportional analogies of geometric figures we concentrate on the Gestalt principles proximity, continuous direction, closure, and similarity. Since these principles refer to single primitive visual elements they are relatively straightforward to implement (principles referring to "external" knowledge like the principle of habit cannot be taken into account by HDTP since a memory would be required). We then could calculate different Gestalts from the formally described geometric figures. Afterwards we applied HDTP to compute the analogies: The analogy making process was executed as follows: The source domain consists of four primitive visual elements. Applying the Gestalt principles, we could compute two different representations of the source domain: following the proximity principle the source domain was formalized as shown in figure 9 (two units consisting of a square and a circle above, while the second unit is a slightly modified iteration of the first unit) and following the continuity principle it was formalized as shown in figure 10 (two squares on the same hight and two circles moving downwards). The representation of the source domain according to certain Gestalt principles takes place before the actual analogy making process. As shown in the following two formalizations, the analogy model HDTP receives as input a source domain already represented as one Gestalt. The target domain, however, is represented simply by its primitive visual elements: the position, shape and colour of element *t1* and *t2*. HDTP re-represents the geometric figure in the target domain depending on the perception of the source domain. Therefore the anti-unification process ends up with a different generalization for the different perceptions.

Source Domain:

```
position(s1,2,2).
shape(s1,square).
colour(s1,white).
transform(s1,[move(p(0,2)),shape,colour],s2).
unit(u1,[s1,s2]).
iter(u1,[move(p(4,0)),approx-y(s2,s1,1)],2).
```

Target Domain:

```
position(t1,2,2).
shape(t1,circle).
colour(t1,grey).
position(t2,2,4).
shape(t2,square).
colour(t2,white).
```

Generalization:

```
position(O1,2,Y).
shape(O1,square).
colour(O1,white).
```

Transfer:

```
transform(O1,[move(p(0,DY)),shape,colour],O2).
unit(u1,[O1,O2]).
iter(u1,[move(p(4,0)),approx-y(O2,O1,1)],2).
```

Figure 9: The formalization of source and target domain and the generalization according to the proximity principle.

In the *proximity case*, element *t2* is aligned to element *s1*, because they have the same shape and colour. Element O1 is the variable representing *s1*, respectively *t2* in the generalization of both theories. Since the y-coordinate of *s1* and *t2* differ, they are anti-unified to the variable Y in the generalization.

The source domain contains additional information on how to construct elements *s3* and *s4* from the starting elements *s1* and *s2*: the element *s2* can be created via moving *s1* along the y-axis and changing its shape and colour. Then, *s1* and *s2* are grouped into one unit. *s3* and *s4* are created via an iteration moved along the x-axis and moving the circle towards the square. The generalized operations to transform the elements and iterate the units are transferred to the target domain. The following list shows the anti-instances and their substitutions in source respectively target domain:

$$O1 \rightarrow s1/t2$$
$$O2 \rightarrow s2/t1$$
$$Y \rightarrow 2/4$$
$$DY \rightarrow 2/{-2}$$

If the *principle of good continuation* is used to structure the source domain, the square and the circle are perceived as separate primitive visual elements. The circle *s3* is created by a transformation "move square *s1* four units along the x-axis". The circle *s4* is created by the transformation "move circle *s2* four units along the x-axis and one unit downwards the y-axis".

Source Domain:

```
position(s1,2,2).
shape(s1,square).
colour(s1,white).

position(s2,2,4).
shape(s2,circle).
colour(s2,grey).

transform(s1,[move(p(4,0))],s3).
transform(s2,[move(p(4,-1))],s4).
```

Target Domain:

```
position(t1,2,2).
shape(t1,circle).
colour(t1,grey).

position(t2,2,4).
shape(t2,square).
colour(t2,white).
```

Generalization:

```
position(O1,2,Y1).
shape(O1,square).
colour(O1,white)

position(O2,2,Y2).
shape(O2,circle).
colour(O2,grey).
```

Transfer:

```
transform(O1,[move(p(4,0))],O3).
transform(O2,[move(p(4,-1))],O4).
```

Figure 10: The formalization of source and target domain and the generalization according to the continuity principle.

HDTP now re-represents the target domain to construct a good alignment and generalization of both theories. The analogy is completed by transferring the transformation operations to the target domain. The following generalizations are created during the anti-unification process:

$$O1 \rightarrow s1/t2$$
$$O2 \rightarrow s2/t1$$
$$Y1 \rightarrow 2/4$$
$$Y2 \rightarrow 4/2$$

We clearly separate between the process of computing different Gestalts perceptions of the geometric figures and the analogy making process itself: the solution to the analogy problem proposed by the human subjects depends on the previously perceived Gestalt, but the process of analogy making is carried out separately.

Besides both mentioned analogies, HDTP comes up with another possible analogy shown in figure 11: instead of moving the circle to the square, the square is moved towards the circle. This solution results from an alignment of visual elements according to their position: the white square at (2,2) is aligned with the grey circle at (2,2), the grey circle at (2,4) with the white square at (2,4) etc. This analogy is possible, however it was not mentioned often by subjects in the pre-test and seems to be less preferred and plausible. Assumedly, because it does not follow any Gestalt principles and moreover contradicts the similarity principle.

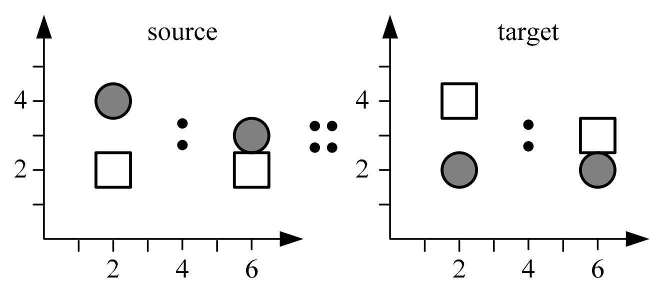


Figure 11: This possible analogy does not follow any Gestalt principles.

Besides the specifications of the source and the target figure, HDTP has some background knowledge on operations applicable to geometric figures, e.g. the spatial relation between two visual elements in the coordinate system can be described via a move operation. Two identical visual elements can be represented via an iteration combined with a move operation: the square *s3* is an iteration of the square *s1*, but moved 4 units to the right. This background knowledge is essential for the automatic re-representation process.

Summary and Future Work

Solving proportional analogy problems between geometric figures is a sophisticated, cognitive process and is often used in intelligence tests. The solution to proportional analogies between geometric figures often differs among humans, because they perceive different conceptualizations of the same figure. This paper shows how Gestalt principles can be used to specify different conceptualizations of figures and how, depending on the conceptualization, the formal analogy model HDTP computes different analogies. HDTP uses first-order logic to represent domains. This allows to represent general rules and do automatic reasoning to infer new representations of the geometric figures. Automatic re-representation is one of the central advantages of HDTP which take effect in computing analogies of geometric figures: the system can compute different Gestalt conceptualizations on-the-fly and use such conceptualizations for analogy making.

The pre-tests evaluating the cognitive plausibility have been promising. However, we are planning a large-scale experiment to investigate psychological preferences among different geometric proportional analogies. Standard IQ tests provide interesting material and benchmark criteria for such experiments. These results have to be included in the heuristics of HDTP. Moreover we will extend the current implementation to enable the computation of Gestalt principles such as similarity and closure.

Acknowledgement

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References

M. Dastani. (1998). *Languages of Perception.* Unpublished PhD Thesis, Universiteit van Amsterdam, Amsterdam.

M. Dastani, & B. Indurkhya. (2001, July 27 - 30, 2001). *Modeling context effect in perceptual domains.* Paper presented at the 3rd International and Interdisciplinary Conference on Modeling and Using Context.

M. Dastani, B. Indurkhya, & R. Scha. (1997). *An algebraic method for solving proportional analogy problems.* Paper presented at the Mind II - Computational Models of Creative Cognition, Dublin, Ireland.

R. C. Dubes. (1993). Cluster analysis and related issues. In C. Chen, L. F. Pan & P. S. P. Wang (Eds.), *Handbook of Pattern Recognition and Computer Vision*: World Scientific.

T. G. Evans. (1969). A program for the solution of a class of geometric-analogy intelligence-test questions. In M. Minsky (Ed.), *Semantic information processing* (pp. 271-353): MIT Press.

H. Gust, K.-U. Kühnberger, & U. Schmid. (2003). *Solving predictive analogy tasks with anti-unification.* Paper presented at the Joint International Conference on Cognitive Science (ICCS/ASCS-2003), Sydney, Australia.

H. Gust, K.-U. Kühnberger, & U. Schmid. (2006). Metaphors and heuristic-driven theory projection (HDTP). *Theoretical Computer Science, 354*(1), 98-117.

D. R. Hofstadter, & J. C. Mitchell. (1995). The copycat project: A model of mental fluidity and analogy-making. In D. R. Hofstadter & F. A. R. group (Eds.), *Fluid Concepts and Creative Analogies* (pp. 205-267): Basic Books.

B. Indurkhya. (1989). *Modes of Analogy.* Paper presented at the International Workshop on Analogical and Inductive Inference (AII) 1989.

B. Indurkhya. (1992). *Metaphor and cognition*. Dodrecht: Kluver.

A. K. Jain, & R. C. Dubes. (1988). *Algorithms for clustering data*. New Jersey, USA: Prenctice-Hall.

K. Koffka. (1935). *Principles of Gestalt Psychology*. New York: Harcourt.

W. Köhler. (1929). *Gestalt Psychology*. New York: Liveright.

S. O'Hara. (1992). *A model of the "redescription" process in the context of geometric proportional analogy problems.* Paper presented at the International Workshop on Analogical and Inductive Inference.

S. O'Hara, & B. Indurkhya. (1995). *Adaptation and re-description in the context of geometric proportional analogies.* Paper presented at the Adaptation of Knowledge for Reuse, Fall Symposium.

G. D. Plotkin. (1970). A note on inductive gneralization. *Machine Intelligence, 5*, 153-163.

G. D. Plotkin. (1971). A further note on inductive generalization. *Machine Intelligence, 6 B*.

M. Wertheimer. (1912). Experimentelle Studien über das Sehen von Bewegung. *Zeitschrift für Psychologie, 61*, 161-265.

M. Wertheimer. (1954). *Productive Thinking*. New York: Harper & Row.